# Positivity of the Kirkwood-Dirac distribution associated with Fourier transform for continuous variable systems

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# The particle on the line = Single mode optical field

- Position space: $\mathbb{R}$, momentum space: $\mathbb{R}$.
- Link between the two via Fourier transform:

$$\hat{\psi}(p) = \langle p|\psi\rangle = \int_{\mathbb{R}} \langle x|\psi\rangle \langle p|x\rangle \, dx = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \psi(x) e^{-ipx} dx$$

- Position basis $(|x\rangle)_{x\in\mathbb{R}}$: $\langle x'|x\rangle = \delta(x'-x)$.
- Momentum basis $(|p\rangle)_{p\in\mathbb{R}}$: $\langle x'|p\rangle = \frac{e^{ipx'}}{\sqrt{2\pi}}$.
- Quadratures: position operator $X$ and momentum operator $P$:

$$X|x\rangle = x|x\rangle \quad \text{and} \quad P|p\rangle = p|p\rangle .$$

# The KD distribution on the real line

**Definition**

For a state $\rho$ on $L^2(\mathbb{R})$, the Kirkwood-Dirac distribution of $\rho$ is

$$\mathrm{KD}_\rho(x,p) = \langle p|x\rangle \langle x|\rho|p\rangle \in \mathbb{C}.$$

- The original distribution introduced by Kirkwood in 1933.
- "Cousin" of the Wigner function (but different !).
- KD distribution of a pure state $|\psi\rangle\langle\psi|$:

$$\mathrm{KD}_\psi(x,p) = \langle p|x\rangle \langle x|\psi\rangle \langle\psi|p\rangle = \frac{e^{-ipx}}{\sqrt{2\pi}}\psi(x)\overline{\widehat{\psi}(p)}$$

# Properties of KD distribution

- Born rule as marginals, Quasiprobability:

$$\int_{\mathbb{R}} \mathrm{KD}_\rho(x,p)dx = \langle p|\, \rho\, |p\rangle\,, \int_{\mathbb{R}} \mathrm{KD}_\rho(x,p)dp = \langle x|\, \rho\, |x\rangle\,,$$

$$\int_{\mathbb{R}^2} \mathrm{KD}_\rho(x,p)dxdp = 1.$$

Goal: Identify the KD-positive states: states $\rho$ for which $\mathrm{KD}_\rho$ is a true probability distribution.

Analogous question for Wigner function: Hudson (1974), Mandilara et.al (2009), Bény et al. (2025), Nicola and Riccardi (2025),...
Applications in metrology: Arvidsson-Shukur et al. (2020).
Applications in classical simulation: Pashayan et al. (2015).

# The KD-positive pure states

We allow normalizable pure states ($\langle\psi|\psi\rangle < +\infty$) and non-normalizable pure states ($\langle\psi|\psi\rangle = +\infty$/undefined).

**Theorem (M.S, 2025)**

*The only pure states $|\psi\rangle$ that have a positive Kirkwood-Dirac distribution are*

- *The position basis* $(|x\rangle)_{x\in\mathbb{R}}$.
- *The momentum basis* $(|p\rangle)_{p\in\mathbb{R}}$.
- *The GKP states* $(|\psi^a_{x_0,p_0}\rangle)_{a>0,x_0\in\mathbb{R},p_0\in\mathbb{R}}$, *where*

$$\langle x|\psi^a_{x_0,p_0}\rangle = \sum_{n\in\mathbb{Z}} e^{inp_0}\delta(x - an + x_0).$$

Very different from Wigner function !

# What about mixed states ?

**Theorem (M.S, 2025)**

*For any mixed state* $\rho \geq 0$, $\mathrm{Tr}(\rho) = 1$, *there exists a set* $V \subset \mathbb{R}^2$ *of strictly positive Lebesgue measure such that*

$$\mathrm{Im}(\mathrm{KD}_\rho(x,p)) \neq 0$$

*for any* $(x,p) \in V$. *In other words* ***there are no KD-positive mixed state*** *for the particle on the line.*

# The particle on the circle (Rotor)

- Position space: the circle $\mathbb{S}^1 \simeq [0,1[$.
- Momentum space: the integers $\mathbb{Z}$.
- Link between the two via Fourier series

$$c_k(\psi) = \langle k|\psi\rangle = \int_0^1 \langle x|\psi\rangle \langle k|x\rangle \, dx = \int_0^1 \psi(x) e^{-2\pi i k x} dx.$$

- Position basis $(|x\rangle)_{x\in[0,1[}$, $\langle x'|x\rangle = \delta(x'-x)$
- Momentum basis $(|k\rangle)_{k\in\mathbb{Z}}$, $\langle x'|k\rangle = e^{2\pi i k x'}$.

# The KD distribution on the circle

**Definition**

For a state $\rho$ on $L^2(\mathbb{S}^1)$, the Kirkwood-Dirac distribution of $\rho$ is

$$\mathrm{KD}_\rho(x,k) = \langle k|x\rangle \langle x|\, \rho\, |k\rangle \,.$$

- Function on phase space $\mathbb{S}^1 \times \mathbb{Z}$.
- Same properties as earlier.

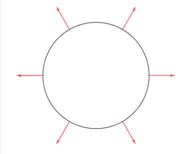
# The KD-positive pure states on the circle

## Theorem (M.S, 2025)

*The only pure states* $|\psi\rangle$ *with a positive KD distribution on the circle are*

- *Circle GKP states (evenly spaced Dirac delta's)* $(|\psi^N_{x_0,k_0}\rangle)_{N\geq 1, x_0\in\mathbb{S}^1, k_0\in\mathbb{Z}}$, *where*

$$\langle x|\psi^N_{x_0,k_0}\rangle = \sum_{k=1}^{N} e^{-2\pi i \frac{k k_0}{N}} \delta\left(x - \frac{k}{N} + x_0\right),$$



- *The momentum basis* $(|k\rangle)_{k\in\mathbb{Z}}$.

# What about mixed states ?

**Theorem (M.S, 2025)**

*A mixed state $\rho \geq 0$, $\mathrm{Tr}(\rho) = 1$ on the circle is KD-positive if and only if it is diagonal in the momentum basis:*

$$\rho = \sum_{k \in \mathbb{Z}} \rho_k \left| k \right\rangle \left\langle k \right| .$$

KD-positive states = only convex combinations of KD-positive pure states.

Again very different from Wigner function !

# References

[1] Stephan De Bièvre, Christopher Langrenez, and Danylo Radchenko. "The Kirkwood-Dirac Representation Associated to the Fourier Transform for Finite Abelian Groups: Positivity". In: *Annales Henri Poincaré* (Sept. 2, 2025).

[2] Matéo Spriet. *Characterizing the Kirkwood-Dirac Positivity on Second Countable LCA Groups*. Sept. 9, 2025. URL: http://arxiv.org/abs/2507.23628. Pre-published.

# Thank you for your attention !