QUICKEST CHANGE-POINT DETECTION WITH CONTINUOUS-VARIABLE QUANTUM STATES

Tiju Cherian John[+*], Christos Gagatsos[*], Boulat A. Bash[*]

[+]BITS Pilani K K Birla Goa Campus, India
[*]Electrical and Computer Engineering Department, University of Arizona, USA

tijucherian@fulbrightmail.org, {cgagatsos,boulat}@arizona.edu

QuiDiQua³
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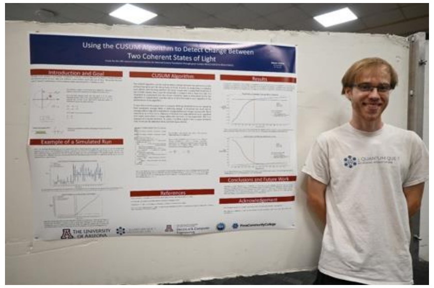
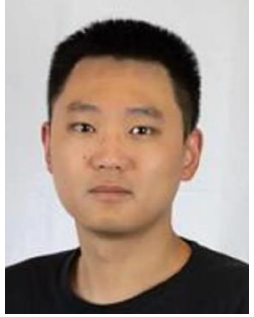


Mason Colony
(Pima County College student, working with Prof. Bash at Univ. of Arizona)

Prof. Boulat Bash
(Univ. of Arizona)

Prof. Tiju Cherian John
(BITS Pilani, ex-Univ. of Arizona Research Scientist)



Qipeng Qian
(Grad student working with CG
on applications of change point detection)

Yihe Wang
(MSc student working with CG
on applications of change point detection)

WHAT IS A CHANGE-POINT DETECTION?

Alice: sends quantum states ρ to Bob but changes to σ at step $\nu + 1$ unknown to Bob.

$\ldots, \sigma, \sigma, \sigma, \rho, \rho, \ldots, \rho, \rho$

Bob: knows that he should expect either ρ or σ. Desires to estimate ν with some confidence.

- Bob measures Alice's output to detect the state change. The number of extra steps Bob takes to declare the change is *latency.*
- The classical version of the problem is well studied, with a wide range of applications, including industrial quality control, onset detection in seismic signal processing, medical diagnostics, and environmental monitoring.
  - In the classical case, instead of sending quantum states, Alice sends i.i.d. samples following a probability distribution P_1 first and changes to P_2 at step ν unknown to Bob.
- Quickest change-point detection was analyzed by Fanizza, Hirche, and Calsamiglia (PRL 131, 020602, 2023) for finite-dimensional input quantum states.
- **Here we study quickest change-point detection for the infinite-dimensional (continuous-variable) input quantum states.**

PERFORMANCE ANALYSIS: CLASSICAL CASE

Strategy and Measurement Outcome

A. Notation

- At step n, Bob receives a sample from the random variable, X_n following an i.i.d. distribution, P_1 for $n \leq \nu$ and P_2 and for $n \geq \nu + 1$.
- The complete history of measurement outcomes up to step n is a vector:

  $\mathbf{X}^n$=$(X_1, X_2, \ldots, X_n)$; $\mathbf{x}^n = (x_1, \ldots, x_n)$
- $\boldsymbol{P_\nu}$ ≔ Probability of a specific sequence of outcomes $\mathbf{x}^n$, given the change happens at step $\nu + 1$.
- $\boldsymbol{E_\nu}$: = Expectation with respect to $\boldsymbol{P_\nu}$ given the change happens at step $\nu + 1$

B. Detection Strategy

- A **change-point detection strategy** S is a decision rule for finding the distribution of the sample received.
- The strategy's output is the **alarm time** T_S which is the step number when the algorithm decides to stop.

Performance Analysis

C. Mean Time to False Alarm

- $\tau_\infty(S)$:=$\boldsymbol{E}_\infty(T_S)$
- An effective strategy must make this value large enough to avoid stopping prematurely.
- **Worst-worst case latency:**

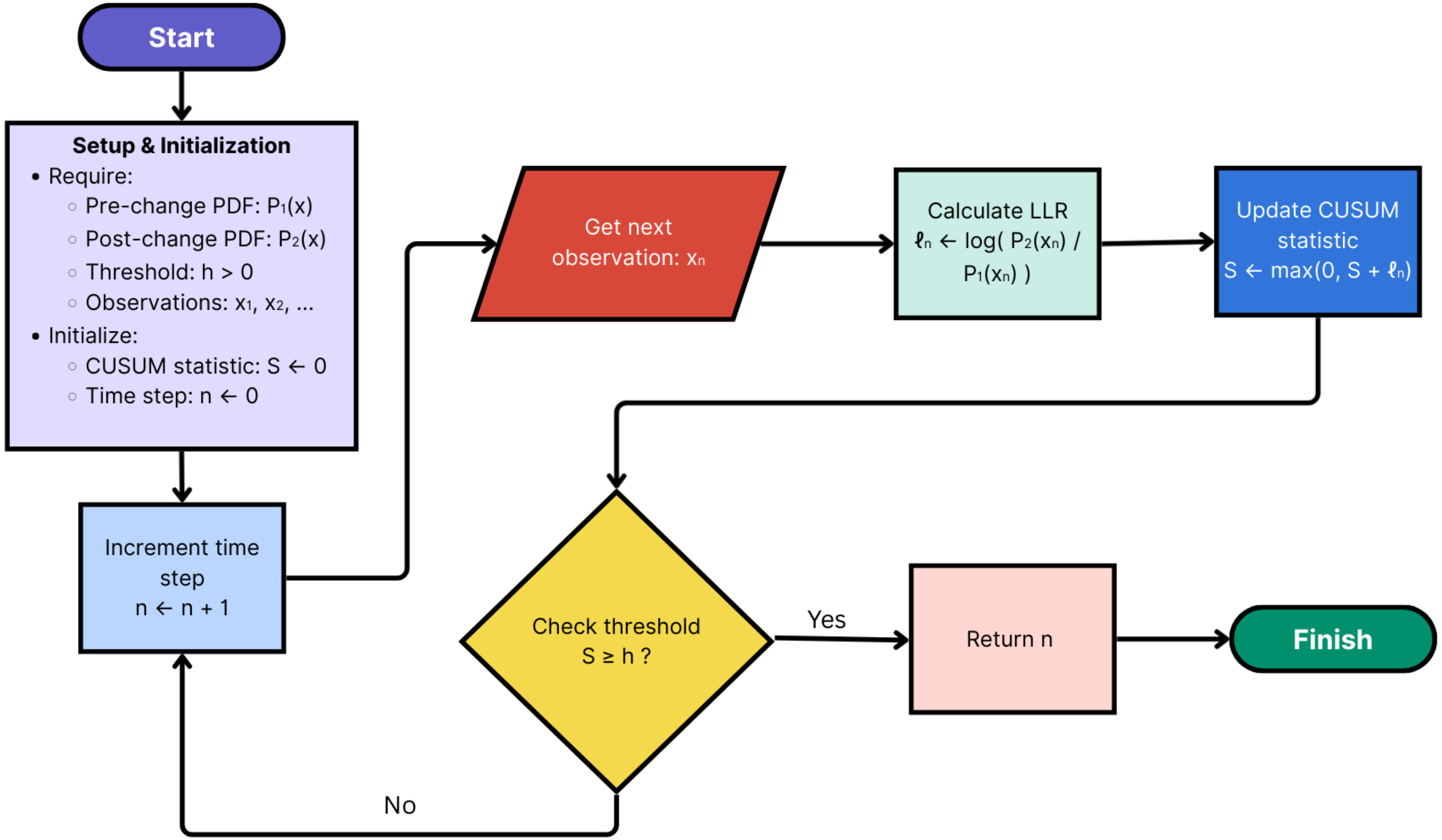
$$\tau(S) = \sup_{\nu \geq 0} \sup_{\{\mathbf{x}^\nu | \boldsymbol{P}_\infty[\boldsymbol{X}^\nu = \mathbf{x}^\nu] > 0\}} \boldsymbol{E}_\nu[T_S - \nu | T_S > \nu, \mathbf{X}^\nu = \mathbf{x}^\nu]$$

- **Lorden's minmax formulation:** The best possible detection performance is the infimum over all strategies, S with $\tau_\infty(S) > 1$

$$\tau_{\min} = \inf_{\{S: \tau_\infty(S) > 1\}} \tau(S)$$

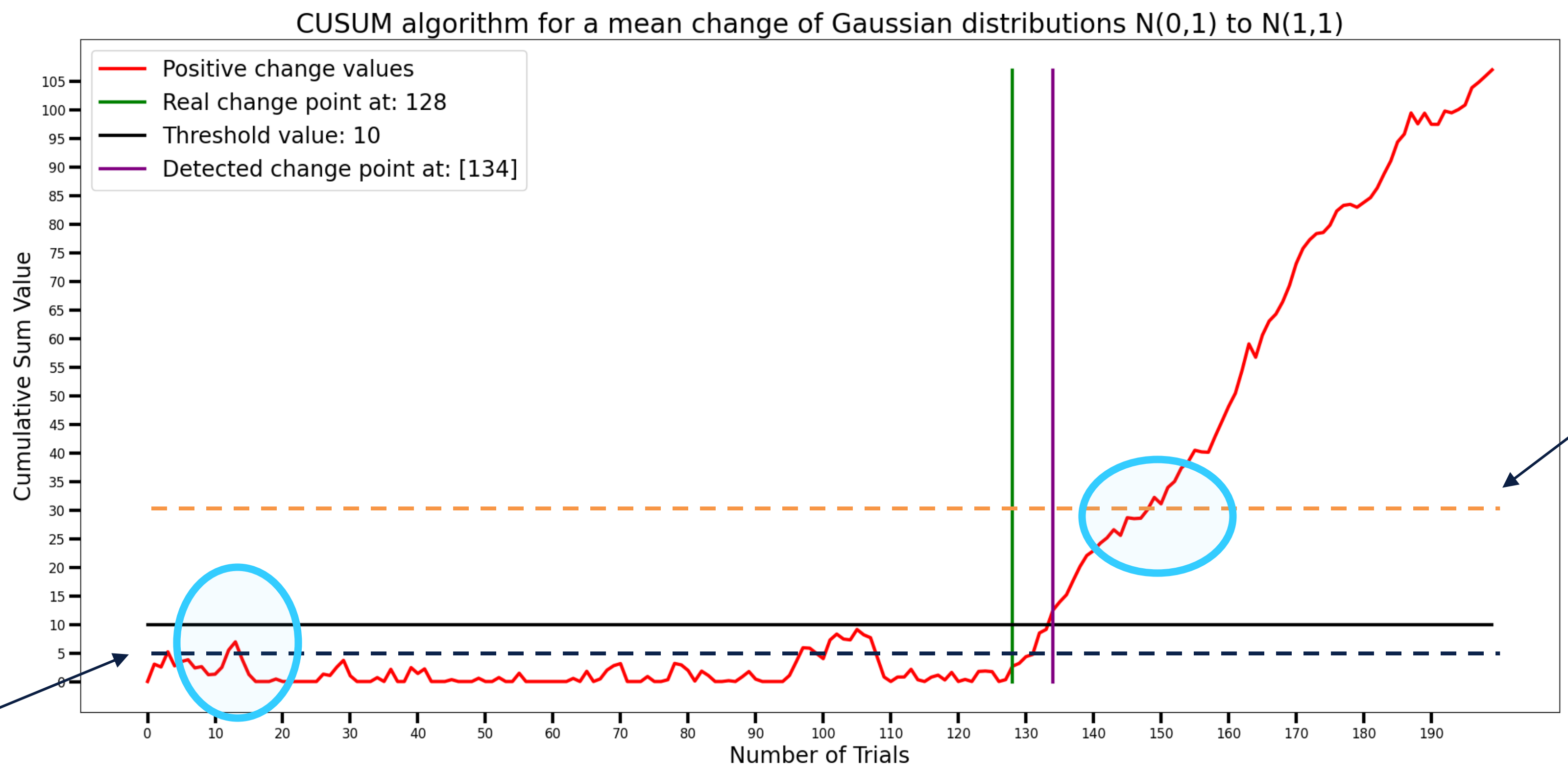
Fact: Cumulative Sum (CUSUM) algorithm achieves the optimum value of the worst-case latency in the **classical case.**

THE CUSUM ALGORITHM



CUSUM CHART

Figure credit: Mason Colony



Threshold setting that results in a late detection.

Threshold setting that results in a false alarm.

Note: we do not consider the problem of setting the threshold here.

OPTIMALITY OF CUSUM

- The mean time to false alarm τ_∞ is the expected value of τ when no change actually occurred (i.e., $\nu = \infty$)
- CUSUM algorithm achieves the optimum value of the worst-case latency

$$\tau_{min} \sim \frac{\log \tau_\infty}{\mathrm{D}(P_2 || P_1)}$$

for large τ_∞, where $\mathrm{D}(P_2 || \, P_1) \; = \int p_2(x) \; \log\left[p_2(x)/p_1(x)\right] dx$ is the relative entropy (Kullback-Leibler divergence) between P_2 and P_1, with corresponding densities $p_2(x)$ and $p_1(x)$.

Sources:

- *E. S. Page. Continuous Inspection Schemes, Biometrika, vol. 41, p. 100, June 1954. (Original proposal of CUSUM algorithm)*
- *G. Lorden. Procedures for reacting to a change in distribution. Annals of Mathematical Statistics, 42(6):1897–1908, Dec. 1971 (Lorden establishes the **asymptotic minimax optimality** of Page's CUSUM procedure)*
- *G. V. Moustakides. Optimal stopping times for detecting changes in distributions. Annals of Statistics, 14(4):1379–1387, Dec. 1986 (the CUSUM procedure is in fact **exactly minimax**)*

QUANTUM CUSUM (QUSUM)

- A **change-point detection strategy** S now has two parts:
  1. A method for performing measurements on the incoming states.
  2. A decision rule for what to do after each measurement.
- Once measurement is fixed, the problem reduces to classical
- Quantum version of Lorden's minmax formulation:

$$\tau_{\min} = \inf_{\{S:\tau_\infty(S)>1\}} \tau(S)$$

  - $\tau(S)$ is still the worst-case latency, defined previously $\tau(S)$ $=\sup_{\nu\geq 0}\sup_{\{\mathbf{x}^\nu|\boldsymbol{P}_\infty[\boldsymbol{X}^\nu=\mathbf{x}^\nu]>0\}} \boldsymbol{E}_\nu[T_S - \nu|\, T_S > \nu, \mathbf{X}^\nu = \mathbf{x}^\nu]$
  - Optimization includes measurements that can jointly process blocks of l states at a time
- Fanizza, Hirche, and Calsamiglia (PRL 131, 020602, 2023): for finite-dimensional σ and ρ:

$$\tau_{\min} = \sim \frac{\log \tau_\infty}{\mathrm{D}(\sigma||\rho)}$$

  - $\mathrm{D}(\sigma||\rho) = Tr[\sigma(\log\sigma - \log\rho)]$ is the quantum relative entropy
- **What about infinite-dimensional (continuous-variable) $\boldsymbol{\sigma}$ and $\boldsymbol{\rho}$?**

ACHIEVABILITY OF QUICKEST CHANGE-POINT DETECTION USING QUSUM

Finite Dimensional case (Fanizza, Hirche, and Calsamiglia (PRL 131, 020602, 2023))	Our Work
After Bob applies a measurement, we have a classical change detection problem obtaining $$\tau(S) \leq \frac{\log\left(\tau_\infty(S)\right)}{D_{\mathcal{M}}(\sigma\|\rho)} + O(1), \text{ as } \tau_\infty(S) \to \infty$$ where the measured relative entropy $D_{\mathcal{M}}(\sigma\|\rho) = \mathrm{D}(P_2\|P_1)$, is the classical relative entropy between outcome probability distributions after applying the measurement $\mathcal{M}$.	Consider infinite-dimensional σ and ρ.

GENERALIZATION OF HAYASHI'S RESULT: STATEMENT

- **Setup:** Let σ and ρ be density operators (quantum states) in an infinite-dimensional separable Hilbert space, H.

- **Conclusion:** Then, there exists a subsequence of natural numbers, $\{l_n\}$, and a sequence of generalized measurements (POVMs), $\{\mathcal{M}^n\}_{n=1}^{\infty}$, on the corresponding multi-copy spaces $H^{\otimes l_n}$, such that the following holds:

$$\lim_{n\to\infty} \frac{D_{\mathcal{M}^n}(\sigma^{\otimes l_n}||\rho^{\otimes l_n})}{l_n} = \mathrm{D}(\sigma||\rho), \qquad \forall \sigma.$$

- **Proof (idea):**

1. Hayashi proved a similar result for finite dimensional states.
2. We reduce our problem to finite-dimensions using certain quantum channels and go back via a two lemmas.

PROOF (SKETCH): GENERALIZATION OF HAYASHI'S RESULT

- **Lemma 1:** Let σ and ρ be two states on an infinite dimensional Hilbert space, H. Then there exists a sequence of finite dimensional Hilbert spaces $\{H_n\}$ and unital quantum channels Φ_n mapping states on H to those on H_n such that

$$\mathrm{D}(\sigma||\rho) = \lim_{n\to\infty} \mathrm{D}(\Phi_n(\sigma)\big|\big|\Phi_n(\rho)).$$

- **Lemma 2:** Let K_1 and K_2 be Hilbert spaces. Let Φ be a unital quantum channel from K_2 to K_1 and let Φ^* denote the dual channel of Φ. If $\mathcal{M}$ is a POVM on K_1 then $\Phi^*(\mathcal{M})$ is a POVM on K_2 and the measured relative entropy w.r.t the POVM $\Phi^*(\mathcal{M})$, satisfies

$$D_{\Phi^*(\mathcal{M})}(\sigma||\rho) = D_{\mathcal{M}}(\Phi(\sigma)\big|\big|\Phi(\rho)),$$

  for all state σ and ρ.

Proofs of lemma are highly technical, involving techniques from the theory of operator algebras.

- Finally, $\mathrm{D}(\sigma||\rho) \underset{\text{Lemma 1}}{\approx} \mathrm{D}(\Phi_n(\sigma)\big|\big|\Phi_n(\rho)) \underset{\text{Hayashi}}{\approx} \frac{D_{\mathcal{M}_n}(\Phi_n(\sigma)^{\otimes l_n}||\Phi_n(\rho)^{\otimes l_n})}{l_n} \underset{\text{Lemma 2}}{=} \frac{D_{\Phi_n^*(\mathcal{M}_n)}(\sigma^{\otimes l_n}||\rho^{\otimes l_n})}{l_n}$ for large n. $\mathcal{M}^n = \Phi_n^*(\mathcal{M}_n)$ is the required sequence of POVMs

Note: The dual relationship between Φ and Φ^* corresponds to the relationship between Schrodinger and Heisenberg representation in quantum mechanics. The channel Φ acts on states while the dual channel Φ^* acts on observables, and, hence, on the elements of a POVM.

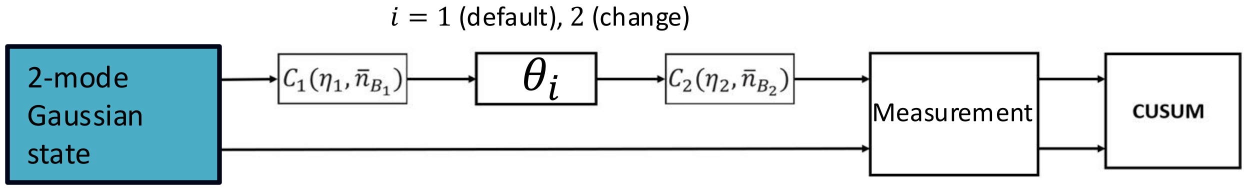
OPTIMALITY OF QUICKEST CHANGE-POINT DETECTION USING QUSUM

Finite Dimensional case (Fanizza et al. PRL, 2023)	Our Work
Optimality: Uses strong converse of Stein's Lemma along with some classical CUSUM analysis	Strong converse of Stein's lemma is not available in the infinite dimensional case. We reduce the problem to the finite dimensional case.
$\tau \geq (1-\epsilon)\,\frac{\log \tau_\infty}{\mathrm{D}(\sigma \| \rho)}\,(1+o(1)),$ as τ_∞ goes to infinity.	• Fix an increasing sequence of finite dimensional subspaces of H. • The projection map from H to H_n give rise to a quantum channel. • Optimal detection delay when restricted to H_n decreases as n increases • **Optimality:** Optimal detection delay restricted to H_n converges in the limit of large n to the optimal detection delay on the full space H. Then the proof of optimality can be concluded using the finite-dimensional result of Fanizza et al.

CONCLUSION

- We showed that fundamental bounds PRL 131, 020602, 2023 apply directly to the broad and technologically significant class of continuous variable (CV) quantum states (arXiv:2504.16259, accepted in IEEE ITW).
  - This includes the quantum states of light that underpin much of quantum optics and form the basis for numerous quantum communication protocols.
  - Thus, we provide a fundamental benchmark against which practical systems (such as monitoring equipment malfunctions, localizing the change in fiber transmission loss, detecting adversaries, etc.) can be assessed and optimized.

- Future work: Elaborate more on the converse, or at least, identify the set of quantum states for which we can provide examples.
- Focus on applications. For example,



Floodlight Illumination: $D^{\mathrm{ASE}}_{\mathrm{(yours)}}(\theta_1\|\theta_2) = \frac{4\,a_{12}^2}{\Delta}\,\sin^2\!\left(\frac{\Delta\theta}{2}\right)\,\left[g(\nu_2) - g(\nu_1)\right]$

QRE for generic 2-mode G state:

$$D(\hat{\rho}(\theta_1)\|\hat{\rho}(\theta_2)) = -\Big(g(n_+) + g(n_-)\Big) + \frac{1}{2}\Big(\ln(n_+^2 - 1/4) + \ln(n_-^2 - 1/4)\Big) + \sum_j \operatorname{arccoth}(2D_{jj})\,\mathrm{Tr}[2iV_1\Omega P_j] + \sum_j \operatorname{arccoth}(2D_{jj}) 2i\delta^T\Omega P_j\delta\Big).$$

Optimization on the state given energy constraints and optimal measurement:

Similar work: Quantum-enhanced change detection and joint communication detection. Saikat Guha, Zihao Gong. Phys. Rev. A 112, 032604 (2025).