Spin coherence scale: operator-ordering sensitivity beyond Heisenberg-Wyel

Aaron Z. Goldberg and Anaelle Hertz

National Research Council of Canada

The NRC headquarters is located on the traditional, unceded territory of the Algonquin Anishinaabe and Mohawk peoples.

Agenda

What is coherence

- Coordination or ambiguity

Quantifying coherence

- From classical to quantum

Quadrature coherence

- Many known properties

Spin coherence

- Desiderata
- Catharsis

The team at NRC (and growing!)

What is coherence?

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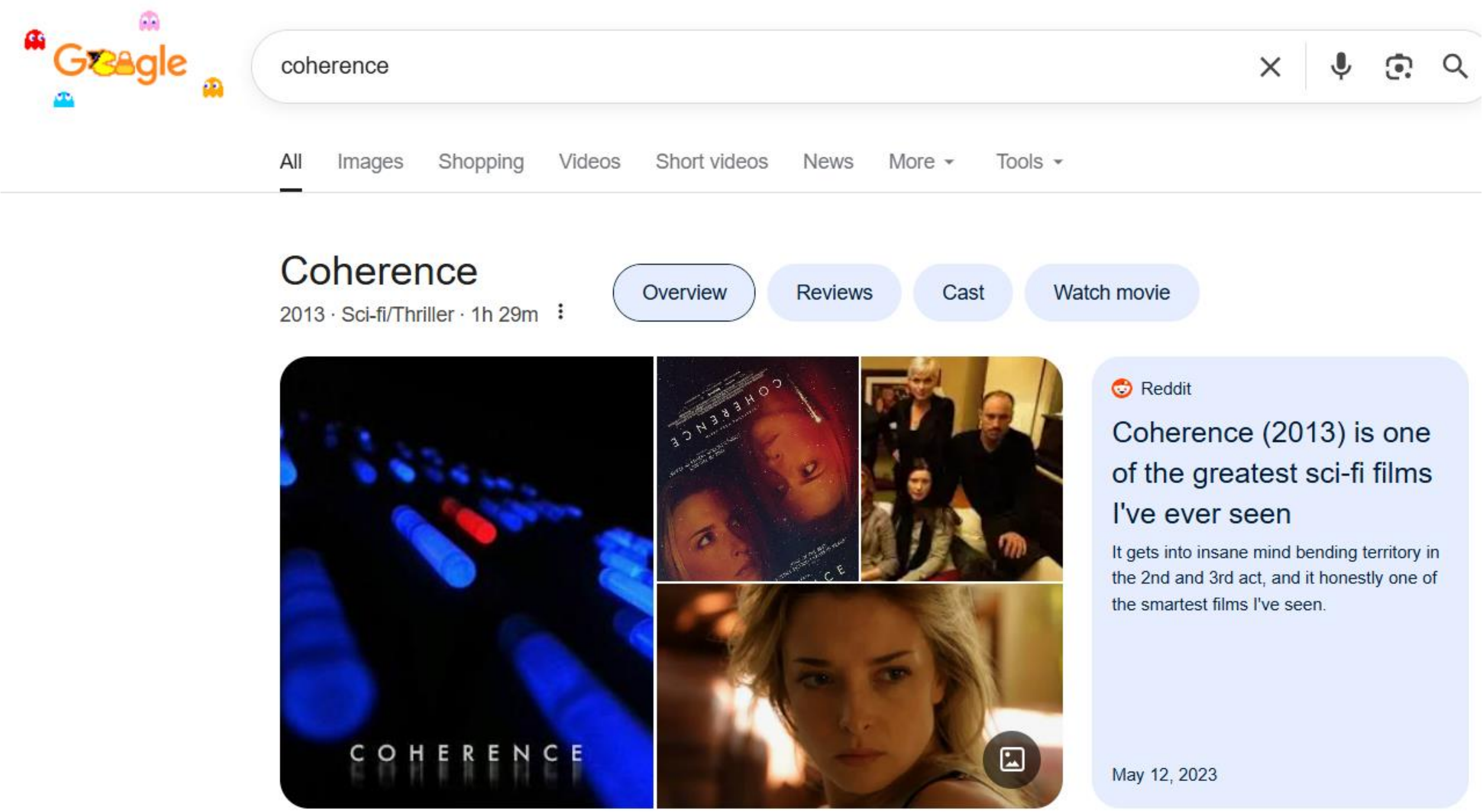

What is coherence?

Dictionary

Definition

Synonyms

Example Sentences

Word History

Rhymes

Entries Near

Show More

Save Word

coherence *noun*

co·her·ence kō-ˈhir-ən(t)s -ˈher-

Synonyms of *coherence* >

1 : the quality or state of cohering: such as

a : systematic or logical connection or consistency

The essay as a whole lacks *coherence*.

b : integration of diverse elements, relationships, or values

"The various parts of this house—discrete in color, in shape, in placement—join together with remarkable *coherence*."
– Paul Goldberger

2 : the property of being coherent

a plan that lacks *coherence*

What is coherence?

Dictionary

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coherent *adjective*

co·her·ent kō-ˈhir-ənt -ˈher-

Synonyms of *coherent* ›

1 a : logically or aesthetically ordered or integrated : **CONSISTENT**

coherent style

a *coherent* argument

b : having clarity or intelligibility : **UNDERSTANDABLE**

a *coherent* person

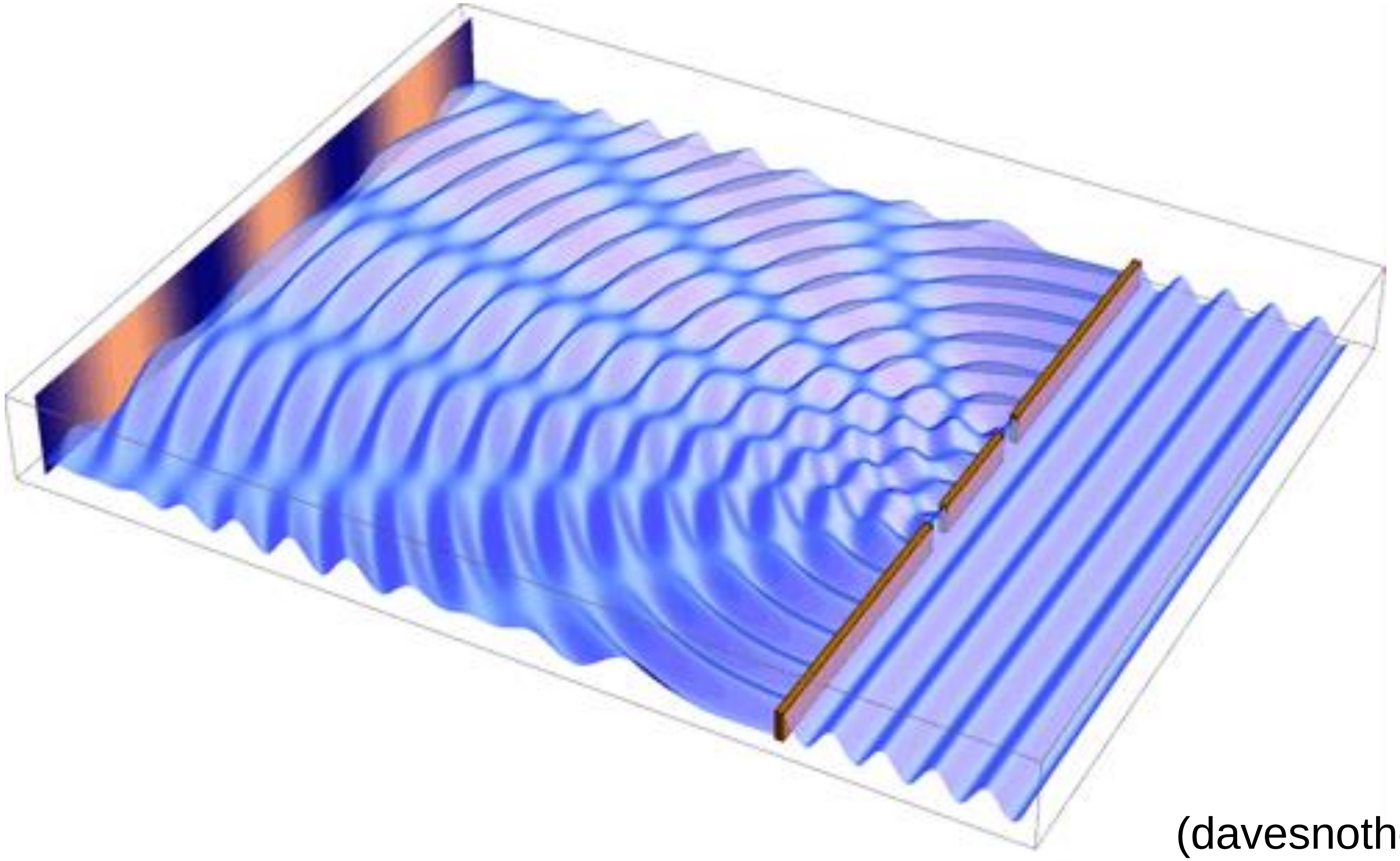
a *coherent* passage

2 : having the quality of holding together or cohering

especially : **COHESIVE, COORDINATED**

a *coherent* plan for action

What is coherence?

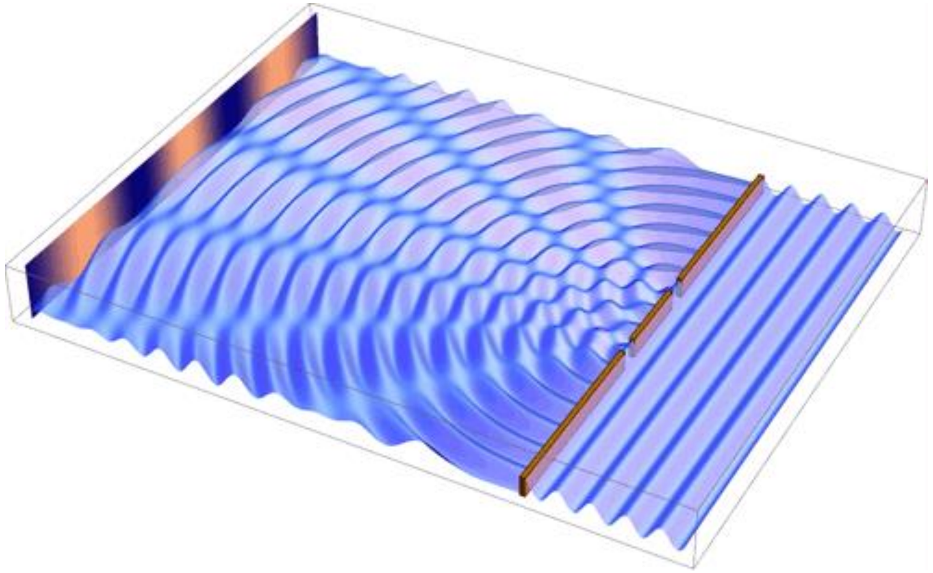


(davesnothere188, tenor)

What is coherence?

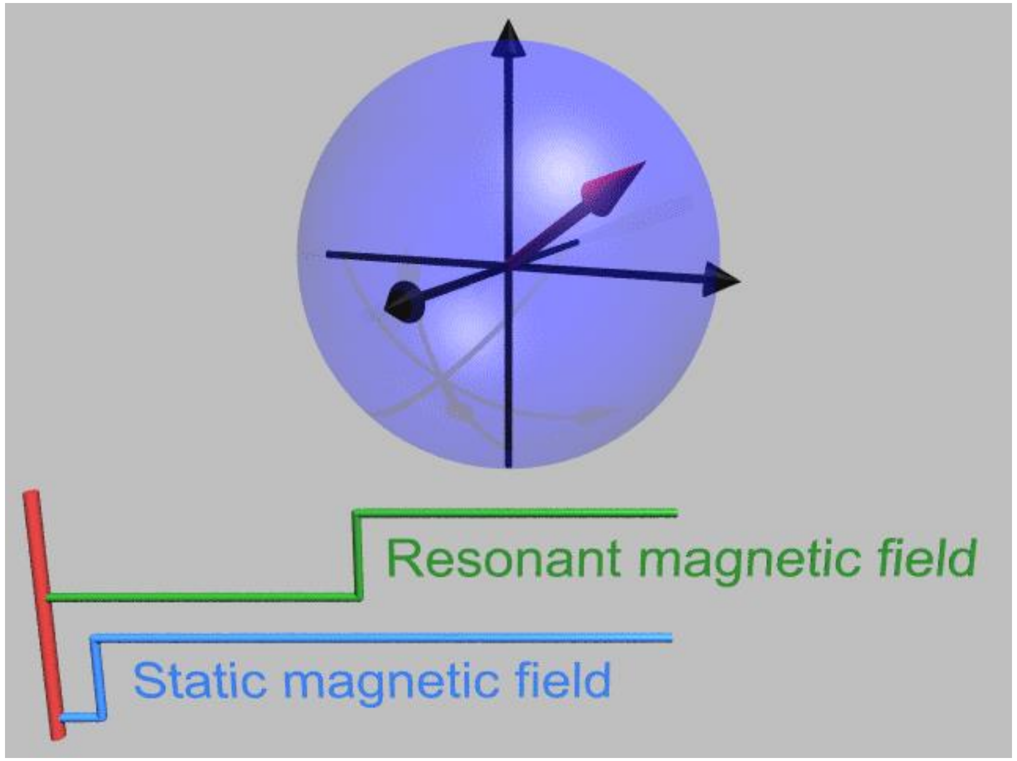
$$\mathrm{Re}\{\exp[\mathrm{i}k(x-x_1)]+\exp[\mathrm{i}k(x-x_2)]\}=\cdots\cos k(x_1-x_2)$$

- The ability of two waves to interfere



Quantum: everything is a wave!

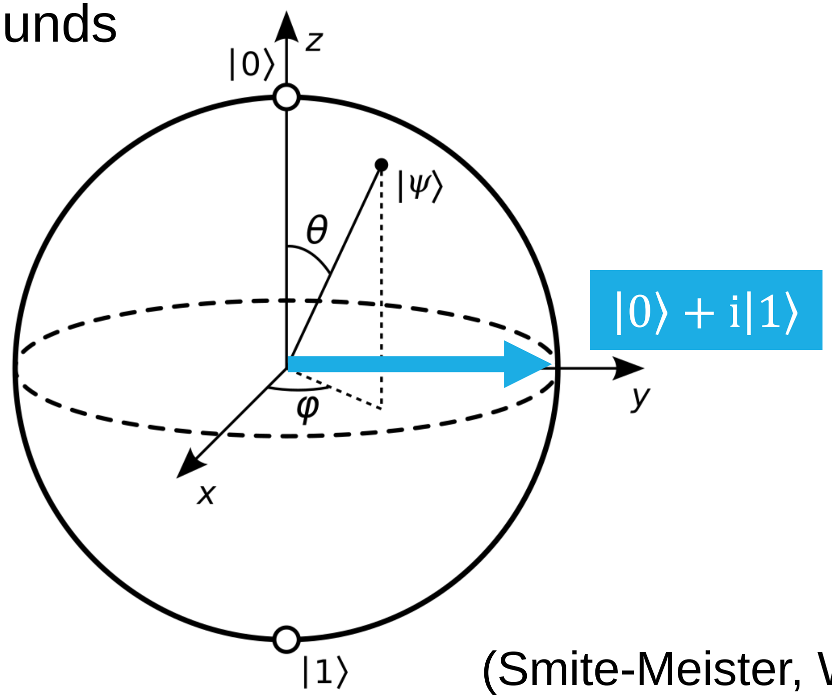
- Coherent spins will precess in a magnetic field, incoherent won’t

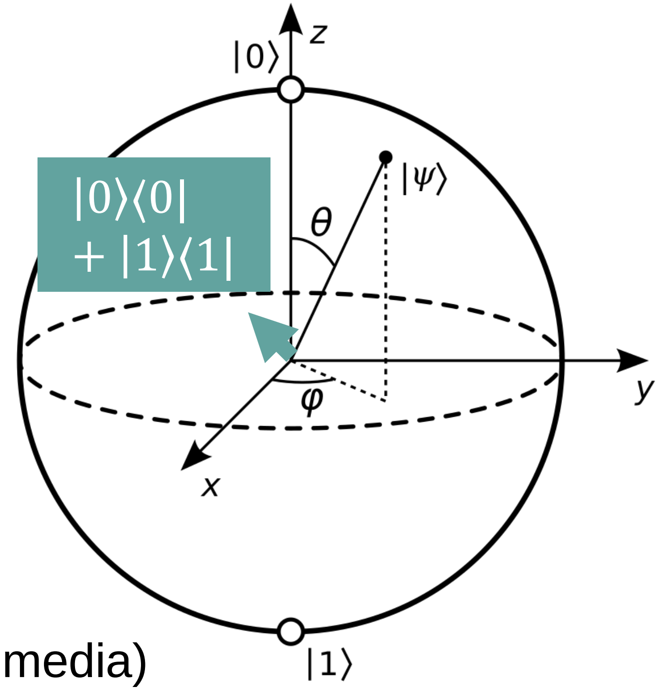


(Gavin W Morley, Wikimedia)

How do you quantify coherence?

- Optics: compare $\langle E^*(x_1)E(x_2)\rangle$ to $\langle |E(x_1)|^2\rangle$, $\langle |E(x_2)|^2\rangle$
  - Spatial, temporal correlations
- Quantum optics: correlations beyond intensity
  - Violate classical bounds
- Spins: can precess
  - More $|0\rangle + \mathrm{e}^{\mathrm{i}\phi}|1\rangle$
  - Less $|0\rangle\langle 0| + |1\rangle\langle 1|$





(Smite-Meister, Wikimedia)

How do you quantify coherence?

- For a qubit things are easy

$$\begin{pmatrix} 1 & e^{i\phi} \\ e^{-i\phi} & 1 \end{pmatrix}/2 \text{ versus } \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}/2$$

  - Magnitude of off-diagonals! Large $|\rho_{01}|$

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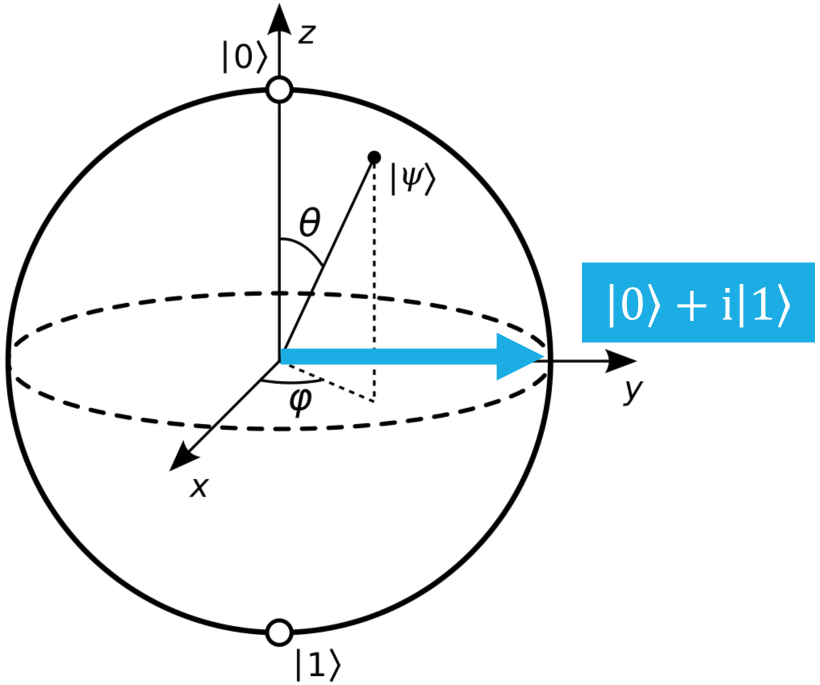
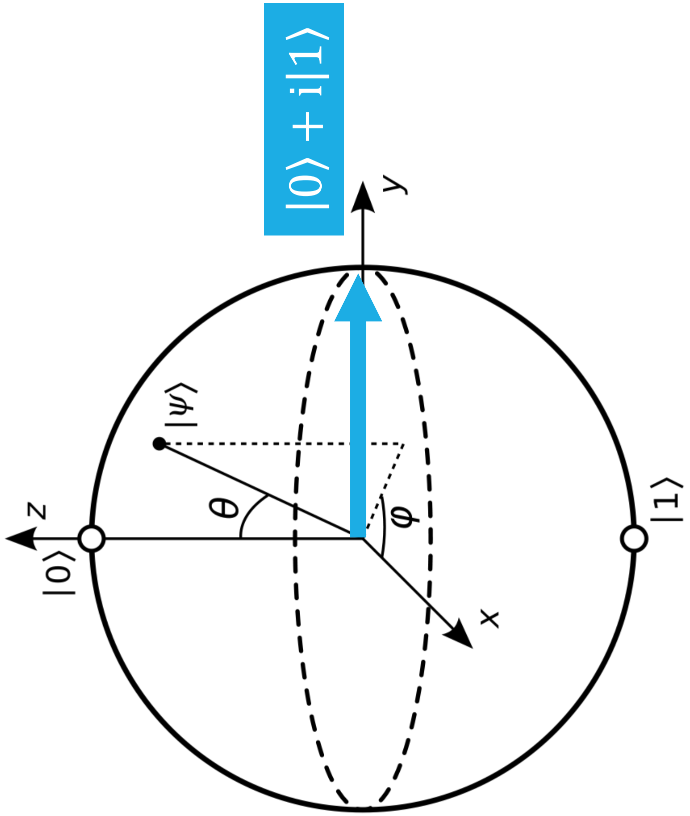
- Not so easy
  - Basis dependent





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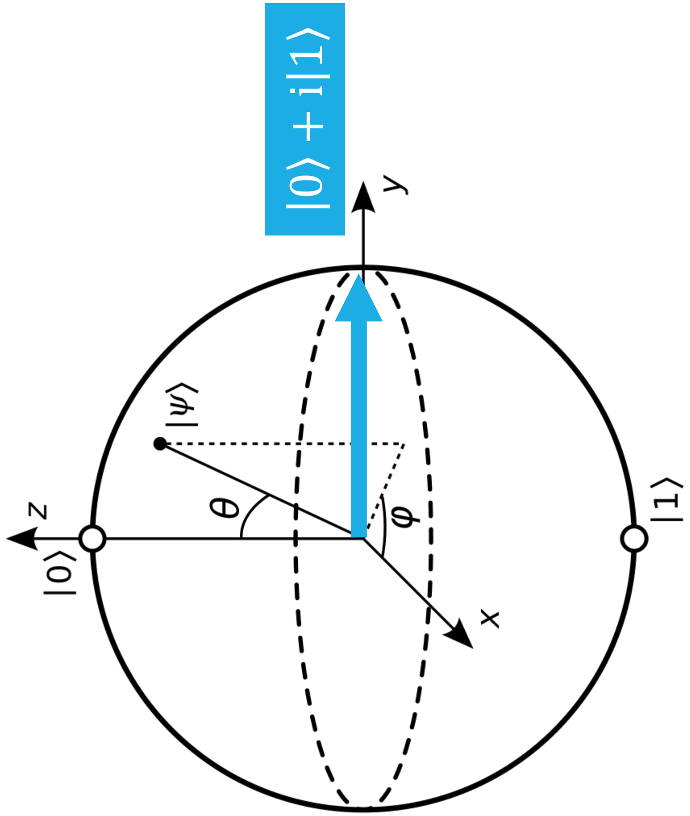
- Goals for a coherence measure
  - Quantify interferability via off diagonalness
  - Coordinate-system agnostic
  - More than two dimensions



Enter the quadrature coherence scale (QCS)

- Single bosonic mode
  - Phase space is position and momentum “quadratures”
    - Infinite dimensions → “continuous variables”
    - Heisenberg-Weyl group generated by x and p with $[x, p] = \mathrm{i}$
  - Superposition of different positions or momenta is uniquely quantum
    - $|x_1\rangle + |x_2\rangle$

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  - Or divergence of Wigner function
  - Or rate of change of purity with loss
- And:
  - Witnesses nonclassicality
  - Bounds distances to classical states
  - Measures quality of displacement sensors (QFI)
  - Convex with loss – probably?

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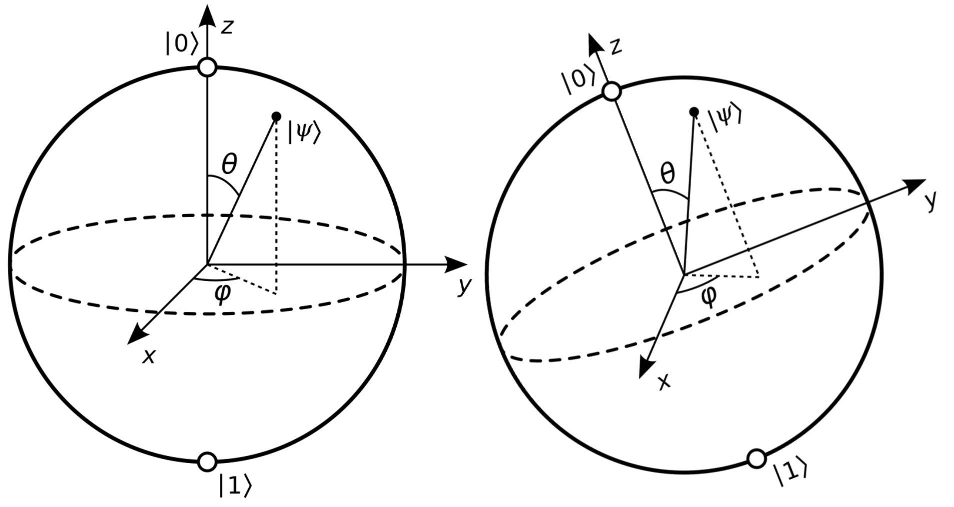
Can we do the same for spins in any finite dimension?

Proposal: spin coherence scale

- From Heisenberg-Weyl to SU(2)
  - Three generators; angular momentum $[J_1, J_2] = \mathrm{i}J_3$
  - Casimir invariant or total spin $J_1^2 + J_2^2 + J_3^2 = J(J+1)$ (qubit has $J = 1/2$)
  - Phase space is a sphere, not a plane
  - Get around by rotations $R(\theta, \boldsymbol{n}) = \exp(i\theta\boldsymbol{J} \cdot \boldsymbol{n})$
- Physics: magnetometry, polarimetry, geodesy



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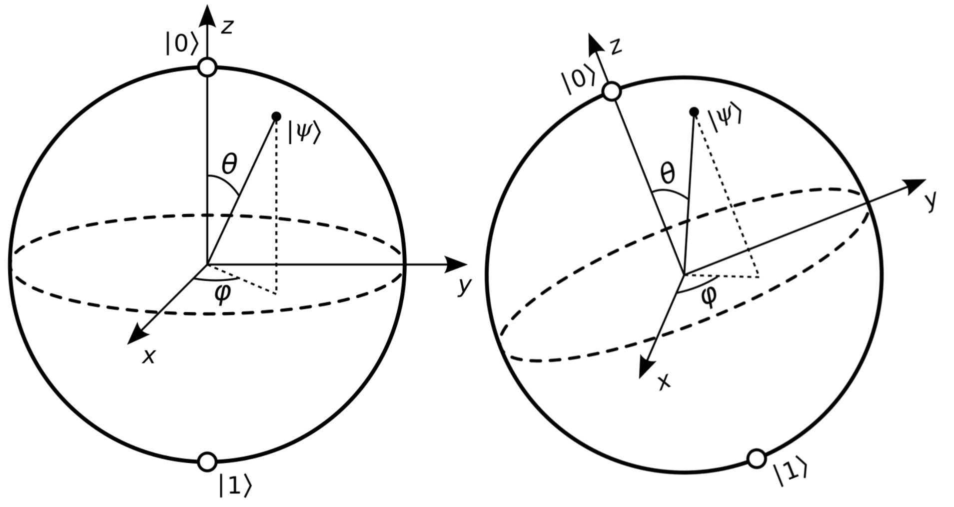
- Physics: magnetometry, polarimetry, geodesy
- Coherence in what basis?
  - Eigenstates $J_3|J,m\rangle = m|J,m\rangle$
  - Or rotate to J_i eigenbasis?
- Noncommutativity of a state $[\rho, J_i]^2$?

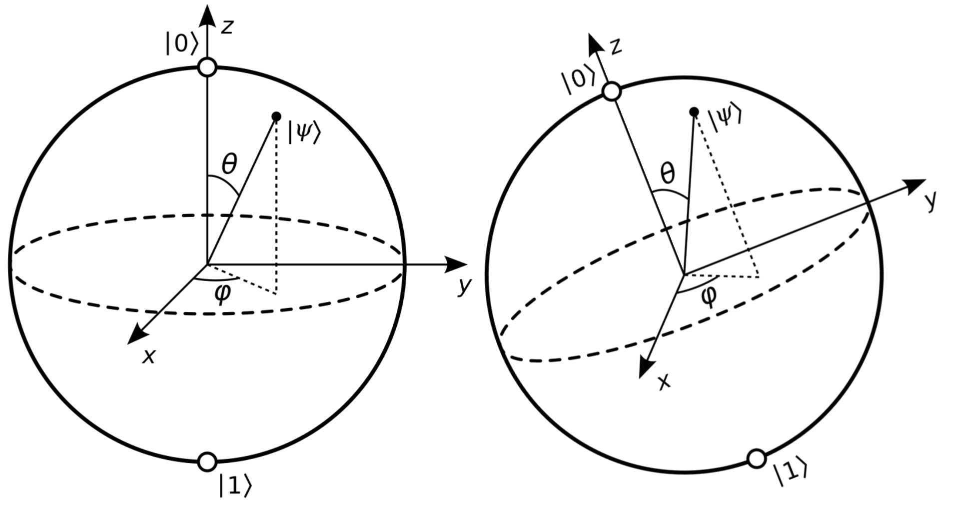


Proposal: spin coherence scale

- Copy the kid's homework

$$\mathcal{A}^2 \propto \sum_{i=1}^{3} \mathrm{Tr}([\rho, J_i]^2) \qquad \mathcal{A}^2 \propto \sum_{mn} \sum_{i=1}^{3} (m-n)^2 |\ {}_i\langle J, m|\rho|J, n\rangle_i\ |^2$$

1. Are they equal?
2. Basis independent?
3. Quantum/classical bound?
4. Nice for pure states? Metrology?
5. Relations to noise? Monotonicity?
6. Quasiprobability distributions?



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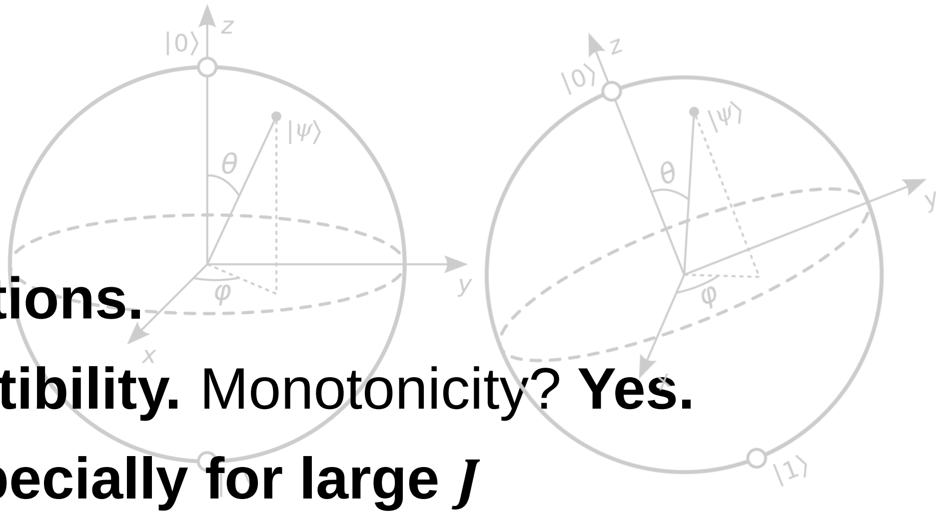
1. Are they equal? **Yes.**
2. Basis independent? **Yes.**
3. Quantum/classical bound? **Yes.**
4. Nice for pure states? **Yes.** Metrology? **Rotations.**
5. Relations to noise? **Depolarization susceptibility.** Monotonicity? **Yes.**
6. Quasiprobability distributions? **You bet. Especially for large *J***



arXiv:2510.09747

Spin coherence scale properties

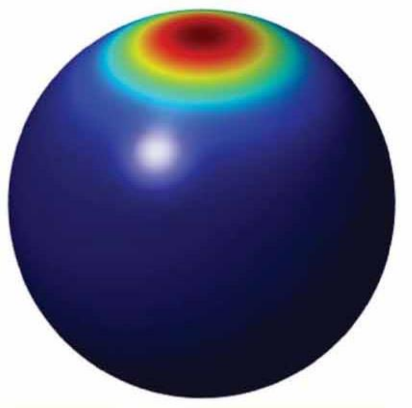
$$\mathcal{A}^2 \propto \sum_{i=1}^{3} \mathrm{Tr}([\rho, J_i]^2) \qquad = \qquad \mathcal{A}^2 \propto \sum_{mn} \sum_{i=1}^{3} (m-n)^2 |\ {}_i\langle J, m|\rho|J, n\rangle_i\ |^2$$

1. Are they equal? **Yes.**
   - Each basis resolves identity $\sum_{m=-J}^{J} |J, m\rangle_i \langle J, m| = \mathbb{I}$
   - Operator ordering sensitivity is the same as coherence scale
2. Basis independent? **Yes.** The vector $\boldsymbol{J}$ rotates like a vector

Spin coherence scale properties

3. Quantum/classical bound? **Yes.** What are the classical states?

- Spin coherent states! (Why we refrain from "SCS")
  - Most localized in phase space
  - Minimize $\Delta^2 J_1 + \Delta^2 J_2 + \Delta^3 J_3$, minimize $\Delta^2 J_i \Delta^2 J_j$
  - Equivalent to $2J$ spin-$\frac{1}{2}$s all aligned
  - $\langle \boldsymbol{\Omega} | \boldsymbol{J} | \boldsymbol{\Omega} \rangle = J \boldsymbol{n}_{\boldsymbol{\Omega}}$
- Mixtures of coherent states
- Prove $\mathcal{A}^2(\sum_k p_k \, |\boldsymbol{\Omega}_k\rangle\langle\boldsymbol{\Omega}_k|) \leq 1$ for any probability distribution p_k
  - Not true if $p_k < 0$
  - Proven by $\sum_{i=1}^{3} |\langle \boldsymbol{\Omega}_k | J_i | \boldsymbol{\Omega}_l \rangle|^2 \geq J^2 |\langle \boldsymbol{\Omega}_k | \boldsymbol{\Omega}_l \rangle|^2$

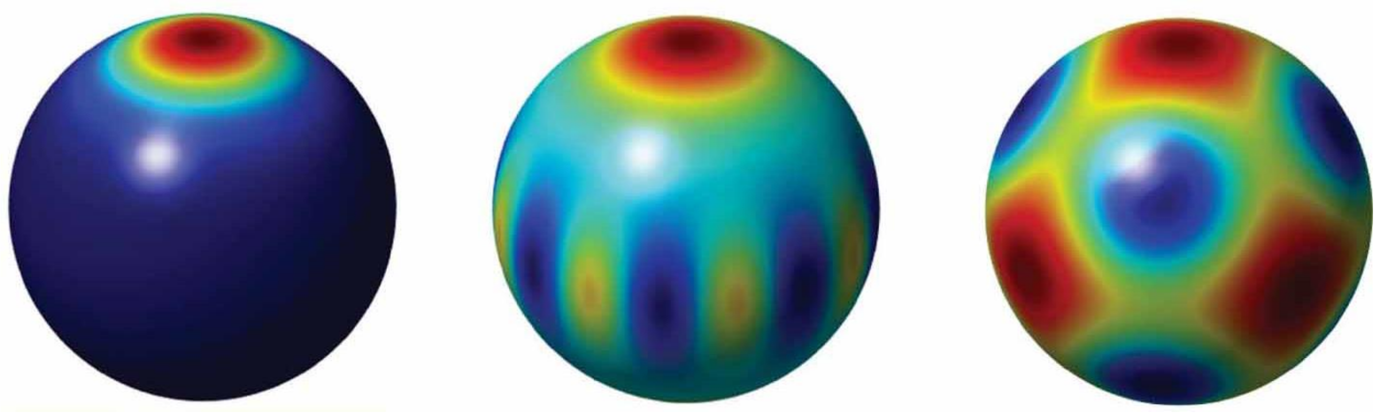


(J. Phys. Phot. 2021)

Spin coherence scale properties

3. Quantum/classical bound? **Yes.** What are the classical states?

- Also bounds distance to set of classical states from both above and below



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$\mathcal{A}^2 = 1$ $\qquad$ $\mathcal{A}^2 \gg 1$ $\qquad$ $\mathcal{A}^2 \gg 1$

Spin coherence scale properties

4. Nice for pure states? **Yes.** Metrology? **Rotations.**

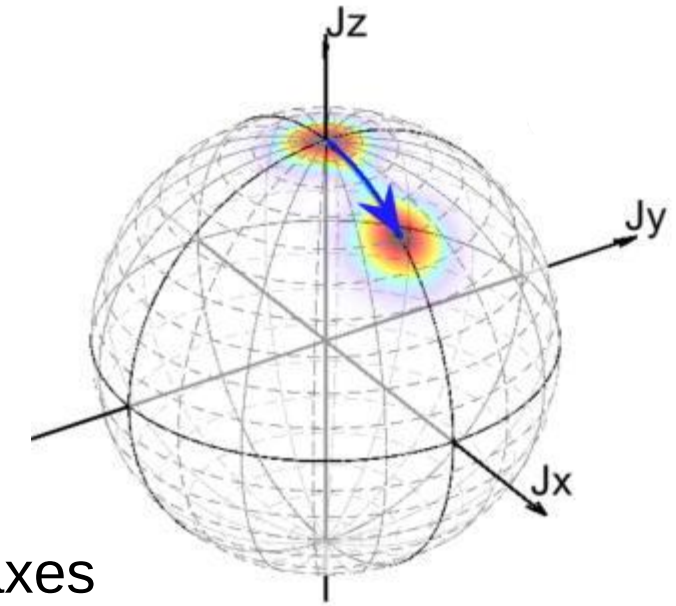
$$\mathcal{A}^2(|\psi\rangle\langle\psi|) = \frac{1}{J}\sum_{i=1}^{3}\Delta^2 J_i$$

- Quantum Fisher information for estimating rotation angle, averaged over all axes is

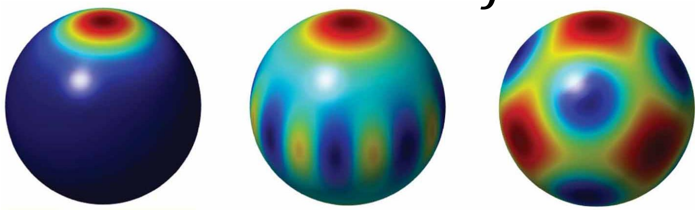
$$\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\mathrm{QFI}(\theta)\; d\Theta \sin\Theta\, d\Phi = \frac{4J}{3}\mathcal{A}^2$$

- Average uncertainty for rotation angle θ, averaged over all axes

$$\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\mathrm{Var}(\theta)\; d\Theta \sin\Theta\, d\Phi \geq \frac{3}{4J\mathcal{A}^2}$$



(Huang *et al.*, App. Phys. Rev. 2024)

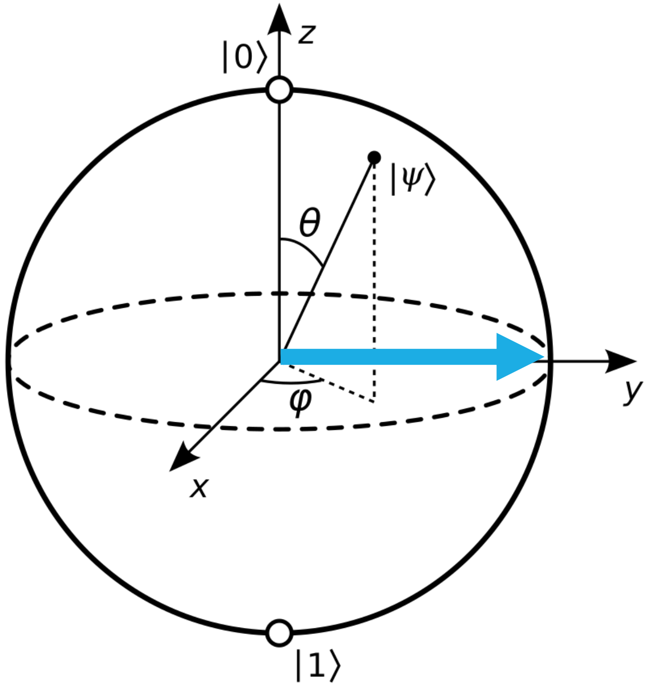


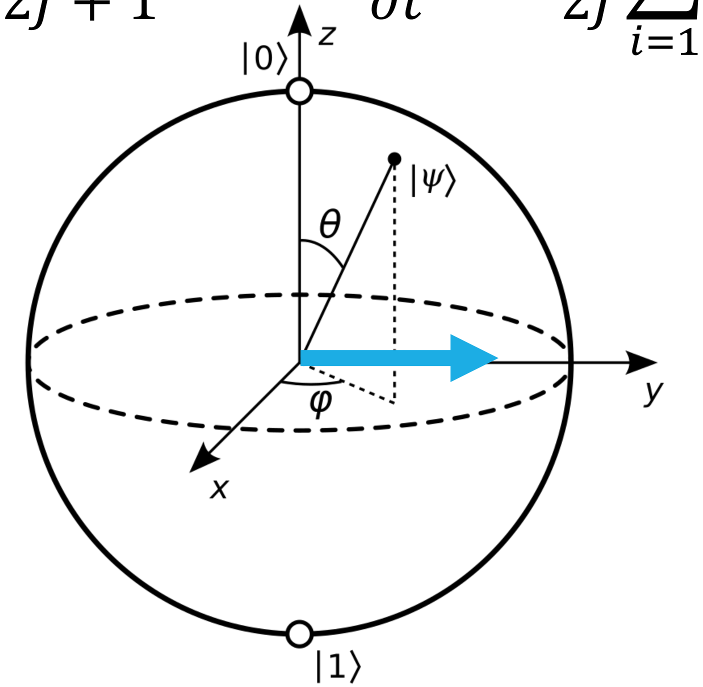
Spin coherence scale properties

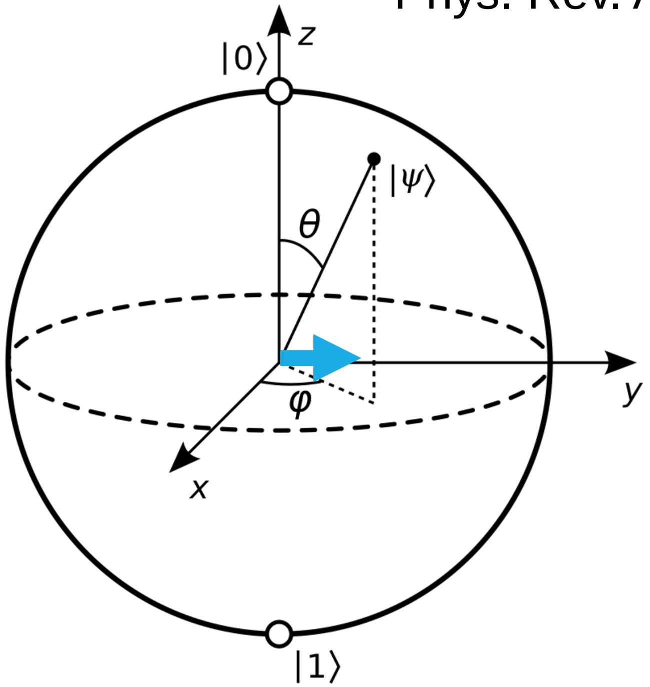
5. Relations to noise? **Depolarization susceptibility.** Monotonicity? **Yes.**

$$\rho \to \frac{\mathbb{I}}{2J+1} \qquad \frac{\partial \rho}{\partial t} = -\frac{1}{2J}\sum_{i=1}^{3}[J_i,[J_i,\rho]]$$

"SU(2)-invariant depolarization" (Rivas & Luis, Phys. Rev. A 2013)







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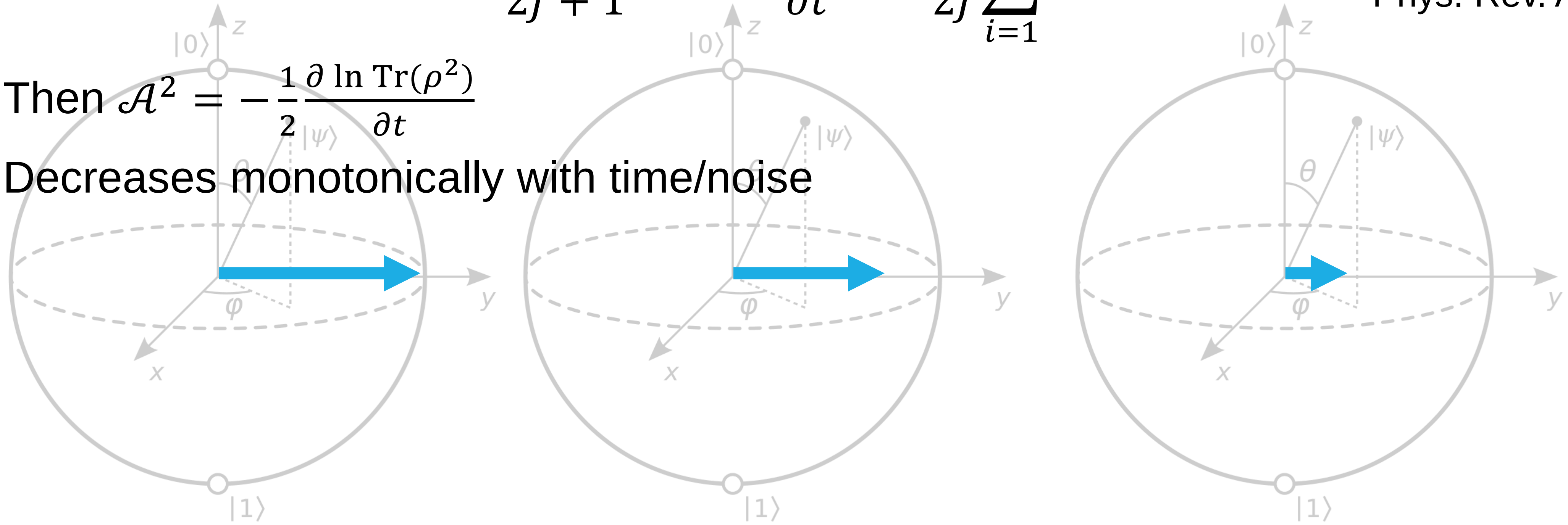
"SU(2)-invariant depolarization" (Rivas & Luis, Phys. Rev. A 2013)

- Then $\mathcal{A}^2 = -\frac{1}{2}\frac{\partial \ln \mathrm{Tr}(\rho^2)}{\partial t}$
- Decreases monotonically with time/noise



Spin coherence scale properties

6. Quasiprobability distributions? **You bet. Especially for large J**

- s-ordered quasiprobability distributions for SU(2)
  - On the sphere
  - From multipoles, spherical harmonics, Clebsch-Gordan coefficients
- Prove for large J:

$$W_{\rho(t)}^{(s)}(\boldsymbol{\Omega}) \approx W_{\rho(0)}^{(s-4t)}(\boldsymbol{\Omega})$$

  - Just like loss susceptibility for Heisenberg-Weyl $W_{\rho(\eta)}^{(s)}(\alpha) = \frac{1}{\eta} W_{\rho(1)}^{\left(1-\frac{s-1}{\eta}\right)}(\alpha/\sqrt{\eta})$

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- Then $\mathcal{A}^2 \approx 4\int W_\rho^{(-s)}(\mathbf{\Omega}) \frac{\partial}{\partial s} W_\rho^{(s)}(\mathbf{\Omega})\, d\mathbf{\Omega} \,/\, \int W_\rho^{(-s)}(\mathbf{\Omega}') W_\rho^{(s)}(\mathbf{\Omega}')\, d\mathbf{\Omega}'$

Is there more?

- It works for more than spins!
- SU(n), at least in the fundamental representation
  - n-mode linear optics
  - Symmetric combinations of n-level systems
  - E.g., boson sampling, three-dimensional polarization, polarimetry

Spin coherence scale: operator-ordering sensitivity beyond Heisenberg-Weyl

Aaron Z. Goldberg, Anaelle Hertz

We introduce the spin coherence scale as a measure of quantum coherence for spin systems, generalizing the quadrature coherence scale (QCS) previously defined for quadrature observables. This SU(2)-invariant measure quantifies the off-diagonal coherences of a quantum state in angular momentum bases, weighted by the classical distinguishability of the superposed states. It serves as a witness of nonclassicality and provides both upper and lower bounds on the Hilbert-Schmidt distance to the set of classical (spin coherent) states. We demonstrate that many hallmark properties of the QCS carry over to the spin setting, including its links to noise susceptibility of a state and moments of quasiprobability distributions. The spin coherence scale has direct implications for quantum metrology in the guise of rotation sensing. We also generalize the framework to SU(n) systems, identifying the unique SU(n)-invariant depolarization channel and outlining a broad, Lie-algebraic approach to defining and characterizing the properties of coherence scale beyond harmonic oscillators.

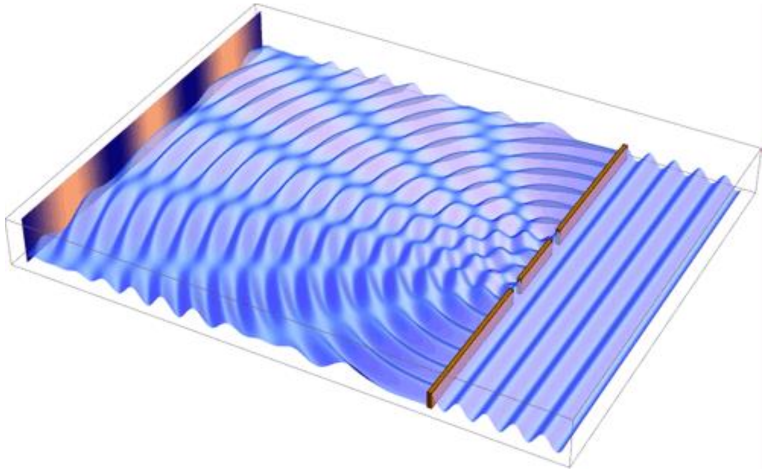
Comments: 12+3 pages; comments always welcome
Subjects: **Quantum Physics (quant-ph)**
Cite as: arXiv:2510.09747 **[quant-ph]**
(or arXiv:2510.09747v1 **[quant-ph]** for this version)
https://doi.org/10.48550/arXiv.2510.09747

Conclusions

- Coherence is about ambiguous properties being simultaneously present

$$\mathcal{A}^2 \propto \sum_{i=1}^{3} \mathrm{Tr}([\rho, J_i]^2) \propto \sum_{mn} \sum_{i=1}^{3} (m-n)^2 |\ {}_i\langle J, m|\rho|J, n\rangle_i\ |^2$$

- Operator ordering sensitivity, macroscopic superpositionness have ties to…
  - Metrology
  - Quantumness
  - Basis-independent coherence measures
  - Noise susceptibility
  - Quasiprobabilities
- …beyond Heisenberg-Weyl! All of these properties are



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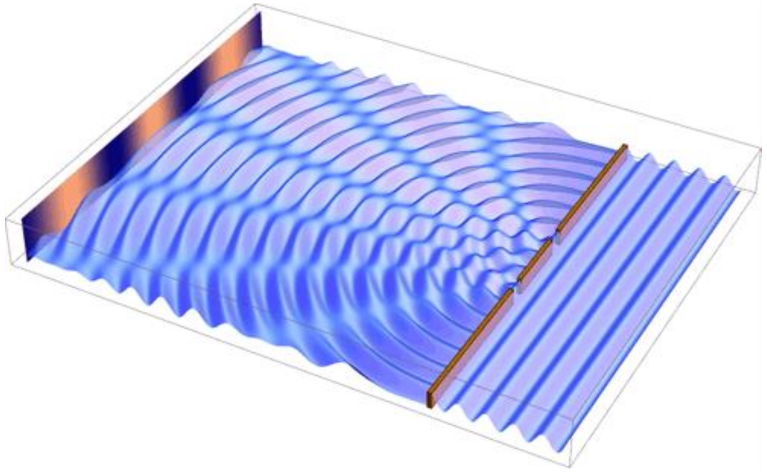

coherent.

$$W_\rho^{(s)}(\Omega) = \sqrt{\frac{4\pi}{2J+1}} \sum_{Kq} \left(C_{JJ,K0}^{JJ} \right)^{-s} \rho_{Kq} Y_{Kq}(\Omega). \quad (5.17)$$

As is clear, the Clebsch-Gordan coefficients are responsible for the transition between the Husimi function

$$W_\rho^{(-1)}(\Omega) = \langle \Omega^{(J)} | \rho | \Omega^{(J)} \rangle \quad (5.18)$$

$$\rho = \sum_{K=0}^{2J} \sum_{q=-K}^{K} \rho_{Kq} T_{Kq}, \tag{5.9}$$

where

$$T_{Kq} = \sqrt{\frac{2K+1}{2J+1}} \sum_{mm'=-J}^{J} C_{Jm,Kq}^{Jm'} |Jm'\rangle\langle Jm| \tag{5.10}$$

$$\int d\Omega W_{\rho(0)}^{(s-4t)}(\Omega) W_{\rho(0)}^{(-s+4t)}(\Omega) \geq \int d\Omega W_{\rho(0)}^{(s-4t')}(\Omega) W_{\rho(0)}^{(-s+4t')}(\Omega)$$

$t \leq t'$, where $\rho(0)$ can be any state.