A grand unification of all dxd discrete Wigner functions

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A Wigner function gap

Continuous domain

- Phase space is the infinite plane,
- There is a unique definition of a Wigner function,
- Satisfies all properties we want (based on the Stratonovich-Weyl criteria).

Phase-point operator, $\hat{A}$

- $\hat{A}$ is Hermitian
- $Tr[\hat{A}] = 1$
- The $\{\hat{A}\}$ are orthogonal

What about *discrete* quantum systems?

Discrete domain

- Phase space is typically a dxd or 2dx2d (**doubled**) grid,
- There is no unique definition of a discrete Wigner function (DWF)
- Different definitions satisfy properties we want for different dimensions.

Odd
Even
Prime
Prime-powered

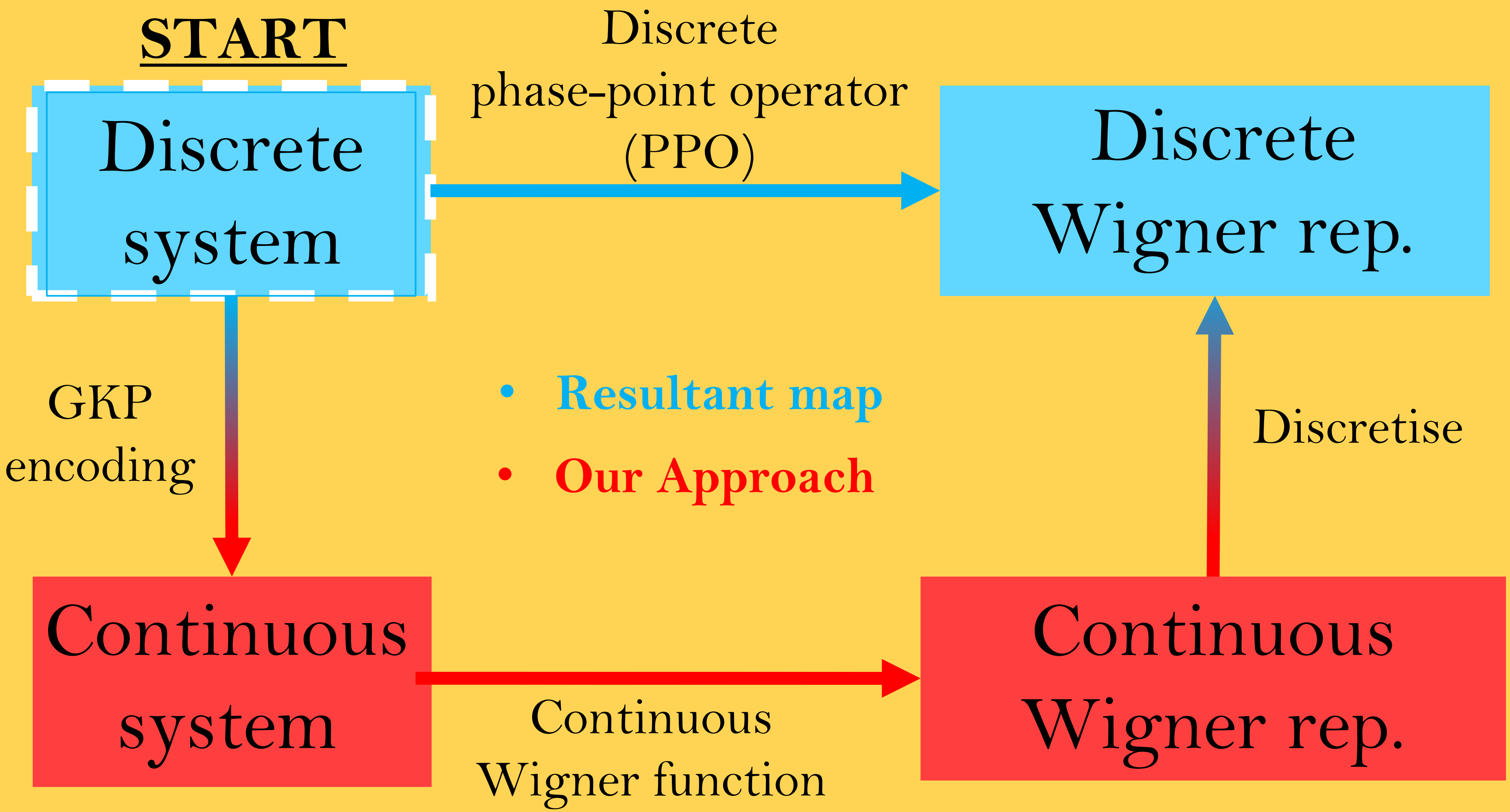
Some Results

- We have developed a general framework that can:

1) Construct **every possible dxd discrete Wigner function (DWF)** for a qudit of a given dimension,
2) Connect all such DWFs through linear maps

- Provide useful tools for system analysis through the DWFs, e.g., resource theories, or simulation of quantum circuits via phase-space methods

A natural, parent function: The doubled discrete Wigner function (DWF)



Daniel Gottesman, Alexei Kitaev, and John Preskill. Encoding a qubit in an oscillator. Physical Review A, 64(1):012310, June 2001

Lucky Antonopoulos

A natural, parent function: The doubled discrete Wigner function (DWF)

$$W_{\hat{O}}^{(2d)}(\mathbf{m}) \coloneqq \frac{1}{2d} Tr\left[\hat{O}\hat{\mathrm{A}}^{(2d)}(\mathbf{m})\right]$$

- This function contains redundancy: 4x the information than what is required for representing d-dimensional systems

$$\hat{A}^{(2d)}(\mathbf{m}) \coloneqq \hat{V}(\mathbf{m})\hat{\mathrm{R}}$$

$$\mathbf{m} \in \mathbb{Z}_{2d}^{2}$$

- PPOs form an overcomplete basis (a frame)

$$\hat{V}(\mathbf{k}) \coloneqq e^{-\frac{2\pi i}{2d}k_1 k_2}\,\hat{\mathrm{Z}}^{k_2}\;\hat{\mathrm{X}}^{k_1}$$

- Can we construct **dxd** DWFs from the doubled one, **free of redundancy**?

- J. Zak, “Doubling feature of the Wigner function: Finite phase space,” Journal of Physics A: Mathematical and Theoretical, vol. 44, no. 34, p. 345305, Aug. 2011.
- M. Saraceno, C. Miquel, and J. P. Paz, “Quantum computers in phase space,” Physical Review A, vol. 65, no. 6, p. 062309, Jun. 2002.
- U. Leonhardt, “Discrete Wigner function and quantum-state tomography,” Physical Review A, vol. 53, no. 5,pp. 2998–3013, May 1996.
- C. Miquel, J. P. Paz, and M. Saraceno, “Quantum computers in phase space,” Physical Review A, vol. 65,no. 6, p. 062309, Jun. 2002.
- Feng and S. Luo, “Connecting Continuous and Discrete Wigner Functions Via GKP Encoding,” International Journal of Theoretical Physics, vol. 63, no. 2, p. 40,Feb. 2024

Constructing to dxd functions

A main result: theorem 1 (stencil theorem)

M-DWF: $$W_{\hat{O}}^{M}(\boldsymbol{\alpha}) := \left(M \star W_{\hat{O}}^{(2d)}\right)(2\boldsymbol{\alpha}) = \frac{1}{d} Tr\left[\hat{A}^{M}(\boldsymbol{\alpha})^{\dagger}\, \hat{O}\right]$$

M-PPO: $$\hat{A}^{M}(\boldsymbol{\alpha}) := \left(M^{*} \star \hat{A}^{(2d)}\right)(2\boldsymbol{\alpha}),$$

Stencil $M : P_{2d} \to \mathbb{C}$

$\boldsymbol{\alpha} \in \mathbb{Z}_{\boldsymbol{d}}^{\boldsymbol{2}}$

- Every valid M-DWF is generated by some stencil M
- Every stencil M generates some valid M-DWF
- **Validity** means that the stencil M (or M-DWF/M-PPO) satisfies criteria we derived based on the discrete analouge of the Stratonovich-Weyl criteria

Constructing to dxd functions

Two stencil examples

- Reduction stencil → dxd DWF valid for **all odd d**

$$\hat{\mathrm{A}}^{M_{RM}}(\boldsymbol{\alpha}) \coloneqq \hat{V}(2\boldsymbol{\alpha})\,\hat{\mathrm{R}}$$

- Leonhardt's,
- Gross',
- Wootters' (prime d>2)

- Coarse-grain stencil → dxd DWF valid for **all even d**

$$\hat{\mathrm{A}}^{M_{CGS}}(\boldsymbol{\alpha}) \coloneqq \frac{1}{2}\sum_{\boldsymbol{b}\in\mathbb{Z}_2^2} \hat{V}(2\boldsymbol{\alpha})\,\hat{\mathrm{R}}$$

- Wootters' (d = 2)
- Chatuverdi's et al (d=2)

U. Leonhardt, Discrete Wigner function and quantum state tomography, Physical Review A 53, 2998 (1996).
D. Gross, Hudson's theorem for finite-dimensional quantum systems, Journal of Mathematical Physics 47, 122107 (2006).
W. K. Wootters, A Wigner-function formulation of finite state quantum mechanics, Annals of Physics 176, 1 (1987).
S. Chaturvedi, N. Mukunda, and R. Simon, Wigner distributions for finite-state systems without redundant phase-point operators, Journal of Physics A: Mathematical and Theoretical 43, 075302 (2010).

Class of unique Ms via projections

$\mathrm{P} : \ell^2(P_{2d}) \to \ell^2(P_{2d})$ $\quad P_{2d} = \mathbb{Z}_{\mathbf{2d}} \times \mathbb{Z}_{\mathbf{2d}}$ $\quad \mathrm{f} : P_{2d} \to \mathbb{C}$

- Many stencils can produce the same M-DWF.
- However, we have identified a unique class of stencils using projections.
- These projections project an arbitrary complex function f onto the space of functions in the image of P
- These projected functions $\bar{\mathrm{f}}$ have the following quasi-periodicity:

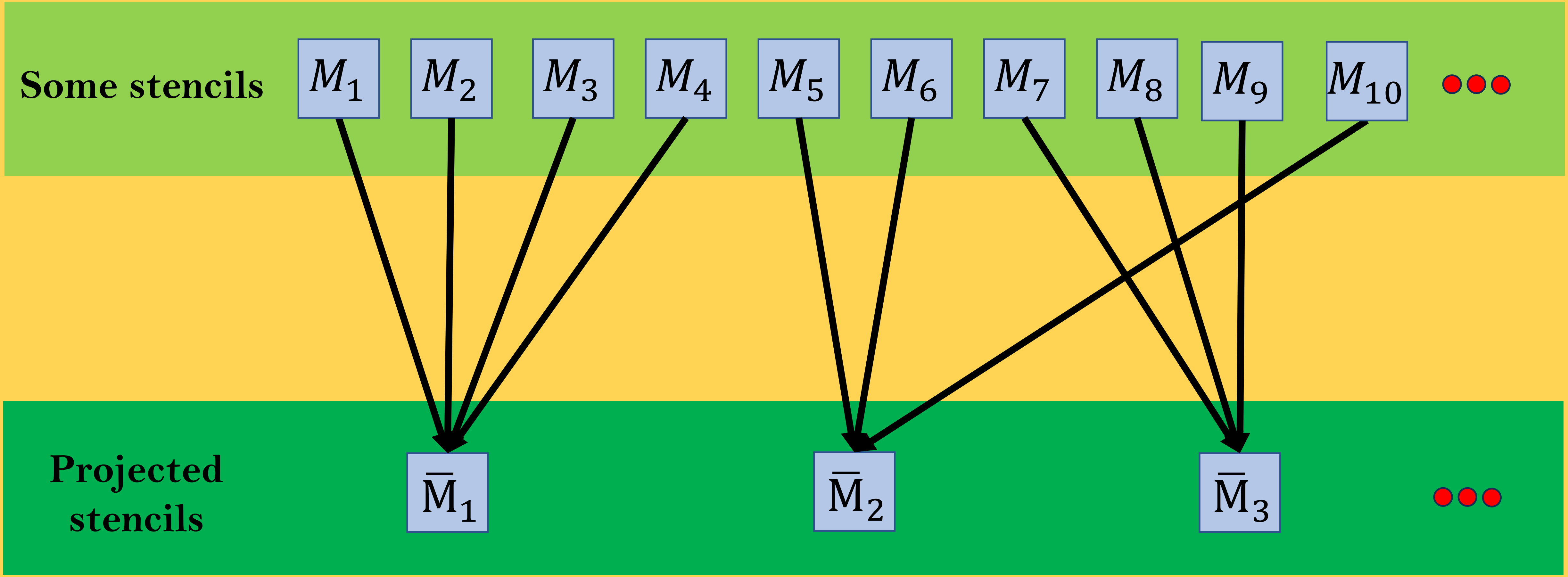
$$\bar{\mathrm{f}}(\mathbf{m} - \mathrm{d}\,\mathbf{b}) = (-1)^{m_1 b_2 - b_2 m_1 - d b_1 b_2}\,\bar{\mathrm{f}}(\mathbf{m})$$

$$\mathbf{b} \in Z_{\mathbb{2}}^{\mathbb{2}}$$

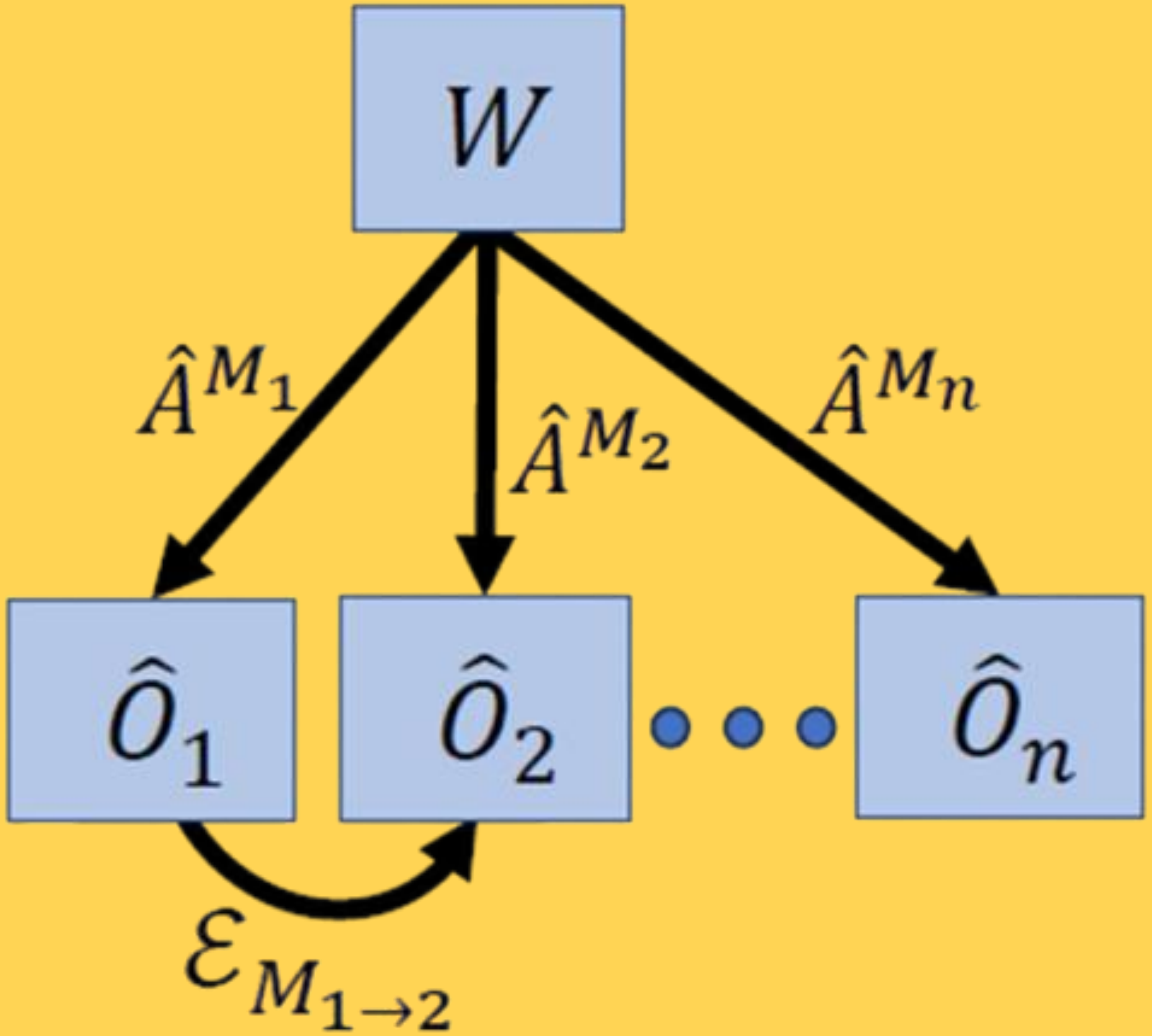
Class of unique Ms via projections

$M : P_{2d} \to \mathbb{C}$

- Example of the uniqueness of stencils

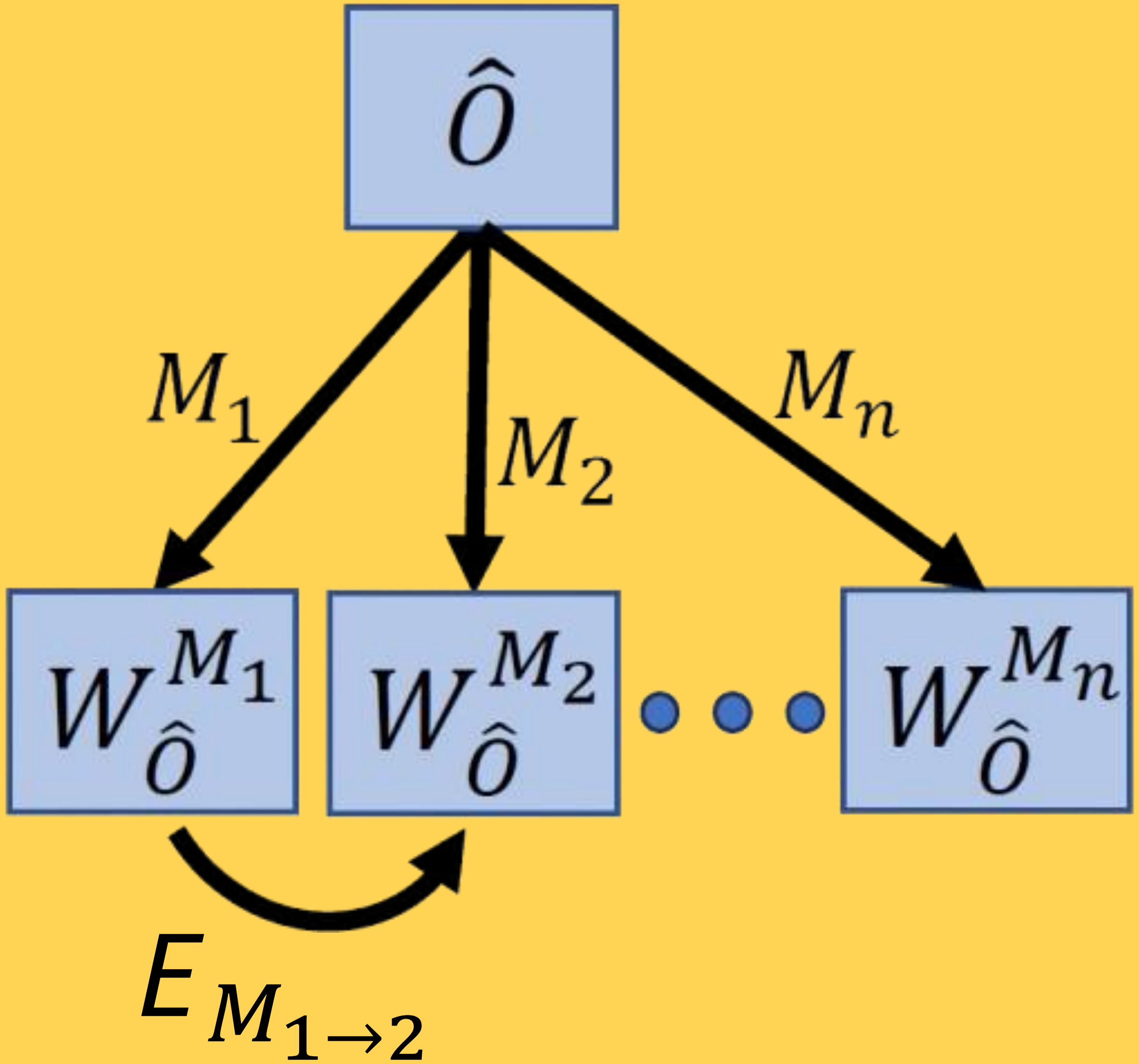


Linear maps



- A function W can represent different operators O by varying the choice of M-PPO
- The O are then related by our linear map ε (acts on operator space)

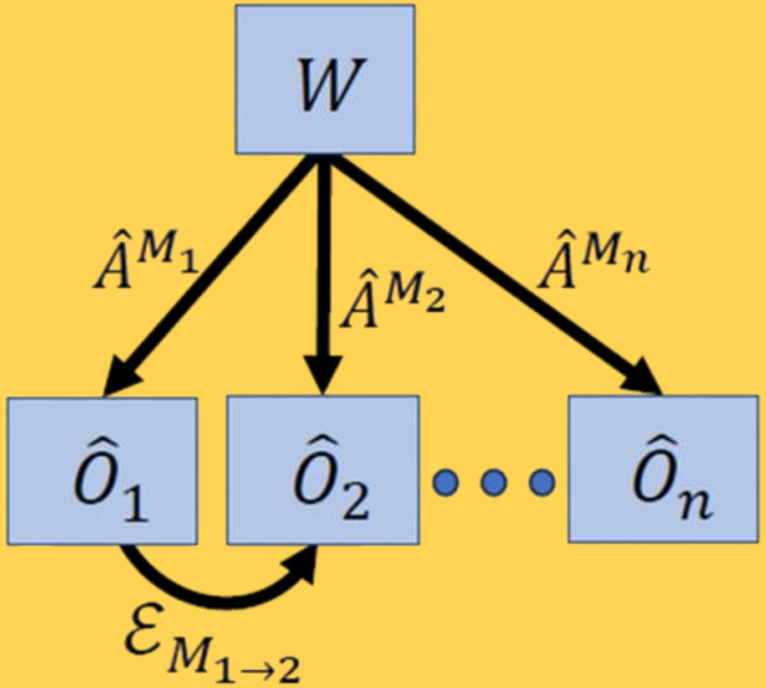
Linear maps

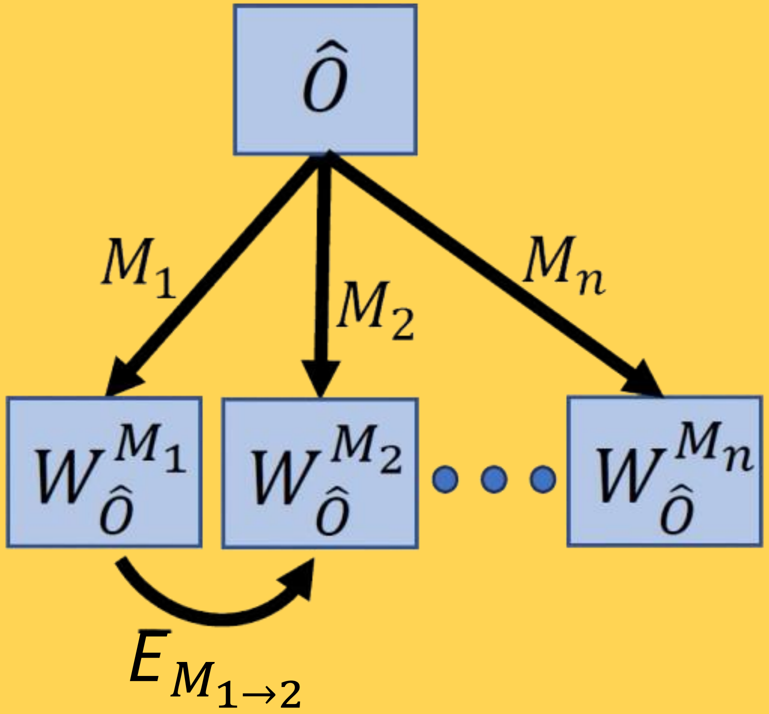


- Each valid DWF W can represent a given operator O, by varying choice of M
- The DWFs are then related by our linear map E (acts on function space)

Linear maps

Theorem 3 in our paper





- These invertible linear maps **further unify all valid DWFs** (for a given dimension) into a **single equivalence class** for a qudit, making difference between DWFs purely representational

- Help identify representation-independent features

- Enable - representation-independent benchmarking of quantum resource measures (e.g., negativity)
  - systematic comparisons of features depending on the chosen stencil M

Summary

- Use a naturally occurring, doubled discrete Wigner function (DWF) as our parent function
- Created a framework for **constructing and unifying all** dxd discrete Wigner functions
- Defined **validity criteria on the stencils---**also in the Fourier domain, related to **discrete Characteristic functions**
- Identified a unique class of stencils M via projections
- Derived a set of **linear maps** that define equivalence classes and enable representation-independent benchmarking
- Future work may explore connecting our stencil framework to other quasidistributions (relaxing one validity criteria leads to all valid discrete **Kirkwood-Dirac** quasidistributions)

Our Paper

https://arxiv.org/abs/2503.09353

Extras: Redundancy

Resultant doubled discrete Wigner function

$$W_{\hat{O}}^{(2d)}(\mathbf{m}) \coloneqq \frac{1}{2d} Tr\left[\hat{O}\hat{\mathrm{A}}^{(2d)}(\mathbf{m})\right]$$

$$\hat{A}^{(2d)}(\mathbf{m}) \coloneqq \hat{V}(\mathbf{m})\hat{\mathrm{R}}$$

$$\mathbf{m} \in \mathbb{Z}_{2d}^2$$

$$\hat{V}(\mathbf{k}) \coloneqq e^{-\frac{2\pi i}{2d}k_1 k_2}\ \hat{\mathrm{Z}}^{k_2}\ \hat{\mathrm{X}}^{k_1}$$

d x d sized phase spaces

- Operators for d-dimensional systems have d^2 parameters
- d^2 unique phase-point operators are needed

2d x 2d (doubled) phase spaces

- There are 4d^2 lattice points and hence 4d^2 phase-point operators (not all unique)
- Doubled phase spaces have a four-fold redundancy
- $A^{(2d)}(\mathbf{m} + \mathrm{d}\ \mathbf{b}) = (-1)^{m_1 b_2 - b_2 m_1 - d b_1 b_2} A^{(2d)}(\mathbf{m})$
- $W_{\hat{O}}^{(2d)}(\mathbf{m} + \mathrm{d}\ \mathbf{b}) = (-1)^{m_1 b_2 - b_2 m_1 - d b_1 b_2} W_{\hat{O}}^{(2d)}(\mathbf{m})$

$$\mathbf{b} \in Z_2^2$$

Extras: Criteria

A1: (Hermiticity) $\hat{A}(\boldsymbol{\alpha}) = \hat{A}(\boldsymbol{\alpha})^\dagger$.

A2: (Normalisation) $\mathrm{Tr}[\hat{A}(\boldsymbol{\alpha})] = 1$.

A3: (Orthogonality) $\mathrm{Tr}\left[\hat{A}(\boldsymbol{\alpha})^\dagger \hat{A}(\boldsymbol{\beta})\right] = d\ \Delta_d[\boldsymbol{\alpha} - \boldsymbol{\beta}]$.

A4: (WHDO covariance) $\hat{V}(\boldsymbol{k})\hat{A}(\boldsymbol{\alpha})\hat{V}^\dagger(\boldsymbol{k}) = \hat{A}(\boldsymbol{\alpha} + \boldsymbol{k})$.

M1: $\bar{M}(\boldsymbol{m})^* = \bar{M}(\boldsymbol{m})$,

M2: $\sum_{\boldsymbol{m}} \bar{M}(\boldsymbol{m}) = 1$,

M3: $(\bar{M} \star \bar{M})(2\boldsymbol{\alpha}) = \Delta_d[\boldsymbol{\alpha}]$.

$\check{\mathrm{M}}1$: $\check{\bar{M}}(\boldsymbol{m})^* = \check{\bar{M}}(-\boldsymbol{m})$,

$\check{\mathrm{M}}2$: $\check{\bar{M}}(\boldsymbol{0}) = \dfrac{1}{2d}$,

$\check{\mathrm{M}}3$: $\left|\check{\bar{M}}(\boldsymbol{m})\right| = \dfrac{1}{2d}$.