Université d'Ottawa | University of Ottawa

Max Planck - University of Ottawa Centre for Extreme and Quantum Photonics

Lundeen Lab Quantum Photonics

Direct Measurement of the Kirkwood-Dirac distribution

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QuiDiQua 2025 Paris

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How do we measure a system's state?

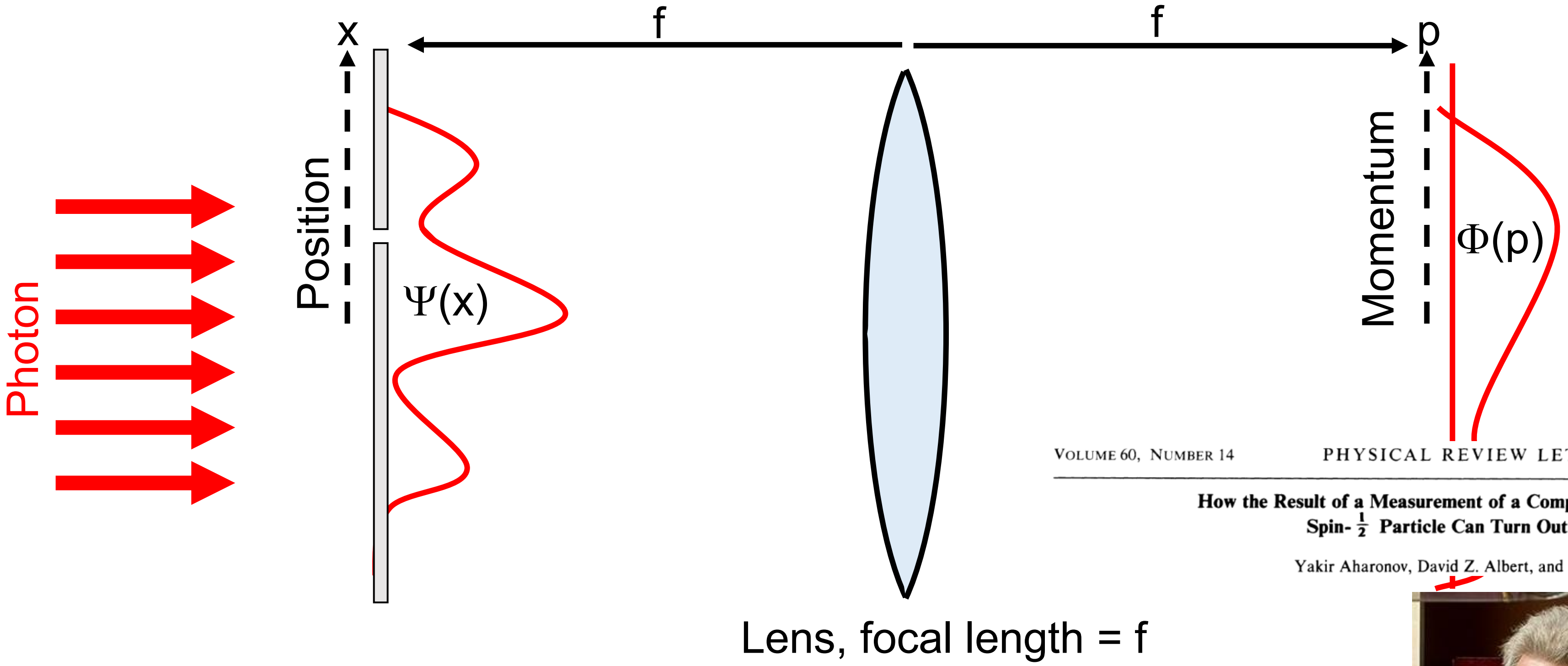
- A classical particle's state is given by its position x and momentum p

- But in quantum physics we have Heisenberg's measurement-disturbance relation:

$$\Delta x\, \Delta p \geq \hbar/2$$

- How can we determine a quantum state?
- This question stumped many great thinkers: Wigner, Fano, Dirac,…

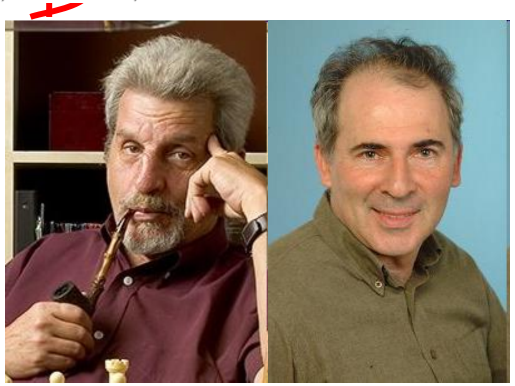
Example of the Heisenberg measurement-disturbance relation



VOLUME 60, NUMBER 14 PHYSICAL REVIEW LETTERS 4 APRIL 1988

How the Result of a Measurement of a Component of the Spin of a Spin-$\frac{1}{2}$ Particle Can Turn Out to be 100

Yakir Aharonov, David Z. Albert, and Lev Vaidman

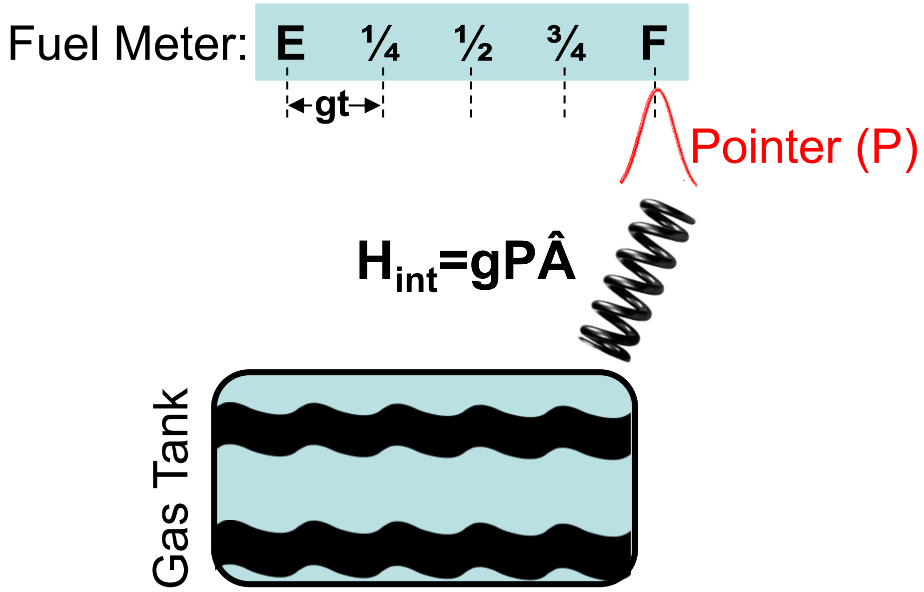


- Measure x precisely and we disturb p so that $\Delta p \rightarrow \infty$
  - ➢ Can not know x and p perfectly at the same time

What if we measure x gently?

Good measurements

Strong Measurement



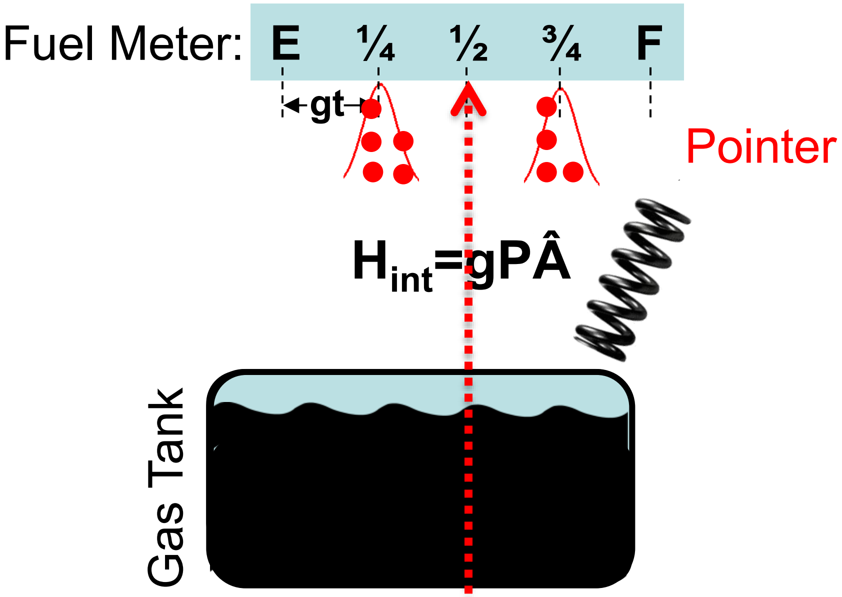
$$\text{System+Pointer}=\sum c_i|a_i\rangle|P_i\rangle$$

Von Neumann: Model both the measured system and the measurement apparatus as quantum systems.

e.g. The pointer needle on a fuel gauge has a wavefunction and so does the gas tank.

Good measurements

Strong Measurement



System=$|¾\rangle$

System+Pointer=$\sum c_i|a_i\rangle|P_i\rangle$

Average Value of **A**:

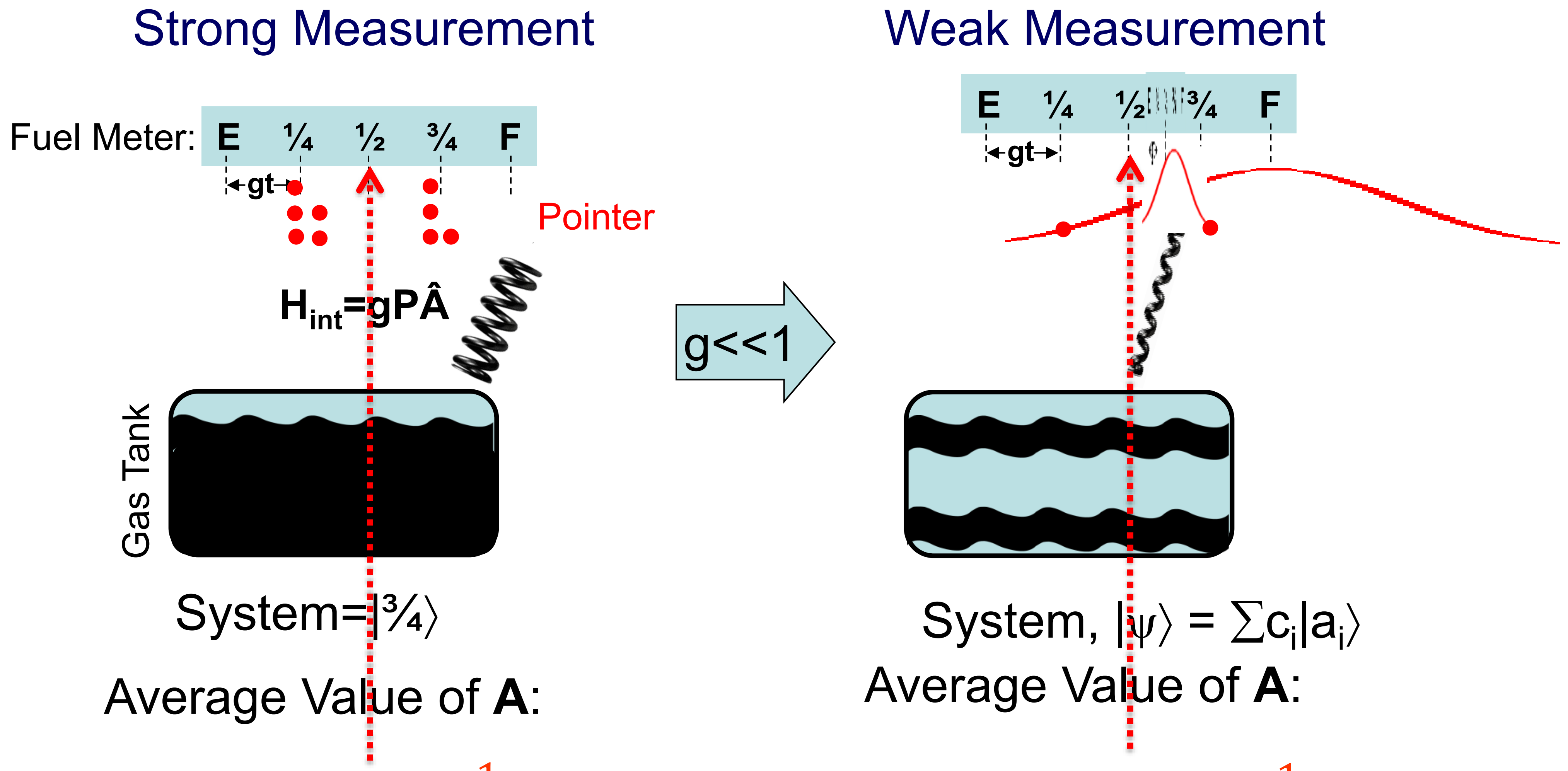
$$\langle\psi|\hat{A}|\psi\rangle = \frac{1}{gt}\langle x_{pointer}\rangle$$

Born Rule!

Von Neumann: Model both the measured system and the measurement apparatus as quantum systems.

e.g. The pointer needle on a fuel gauge has a wavefunction and so does the gas tank.

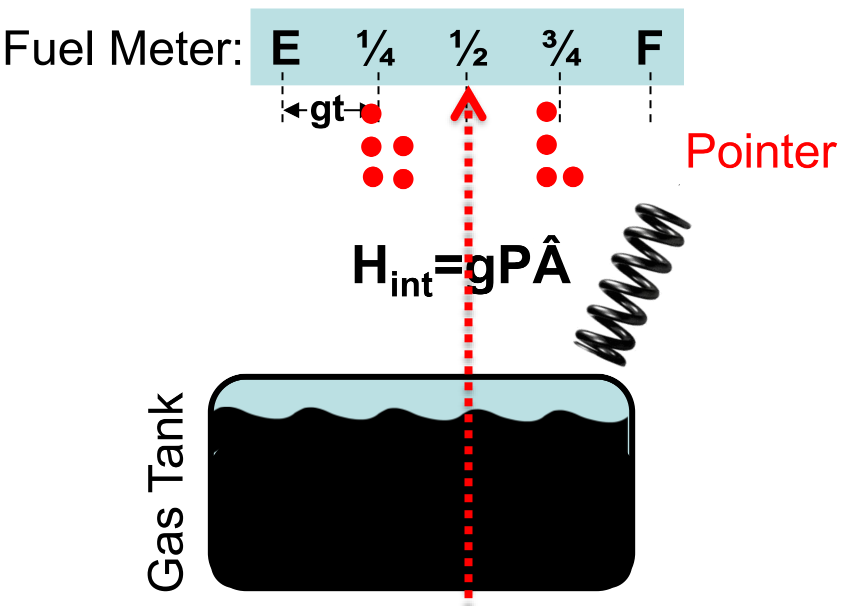
Weak Quantum Measurement



$$\langle\psi|\hat{A}|\psi\rangle = \frac{1}{gt}\langle x_{pointer}\rangle \qquad \langle\psi|\hat{A}|\psi\rangle = \frac{1}{gt}\langle x_{pointer}\rangle$$

Weak Measurement

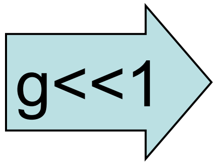
Strong Measurement



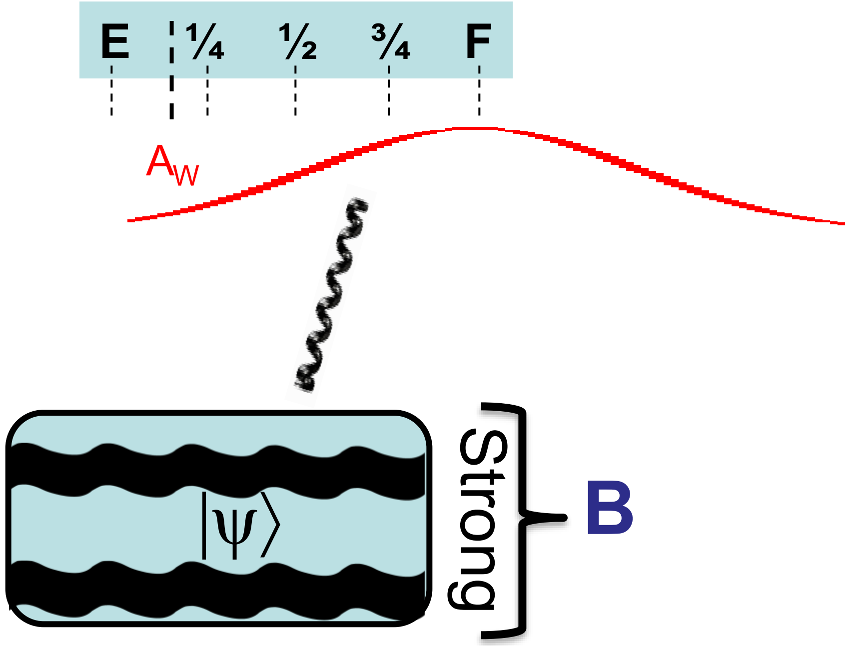
System=|¾⟩

Average Value of **A**:

$$\langle\psi|\hat{A}|\psi\rangle = \frac{1}{gt}\langle x_{pointer}\rangle$$



Weak Measurement



In the cases where result of **B** is b

Average Value of **A** ≡ *Weak Value:*

$$A_W = \frac{\langle b|A|\psi\rangle}{\langle b|\psi\rangle}$$

$$A_W = \frac{1}{gt}\left(\langle x\rangle + i\frac{2\sigma^2}{\hbar}\langle p\rangle\right)$$

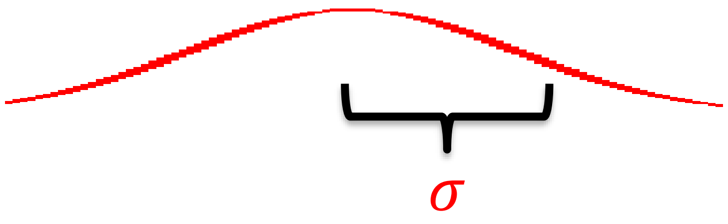
Real part of A_W is the position shift of the pointer

Imaginary part of A_w is the momentum shift of the pointer

The weak value and the lowering operator

Average shift of the pointer when B=b

$$A_W = \frac{2\sigma}{gt}\left(\frac{1}{2\sigma}\langle x\rangle_{Pointer} + i\frac{\sigma}{\hbar}\langle p\rangle_{Pointer}\right)$$



Lundeen & Resch, Phys. Lett. A 334 (2005) 337–344

Harmonic oscillator lowering operator

$$a = \frac{1}{2\sigma}\langle x\rangle + i\frac{\sigma}{\hbar}\langle p\rangle$$

The weak value is proportional to the lowering operater a:

$$A_W = \frac{2\sigma}{gt}\langle a\rangle_{Pointer}$$

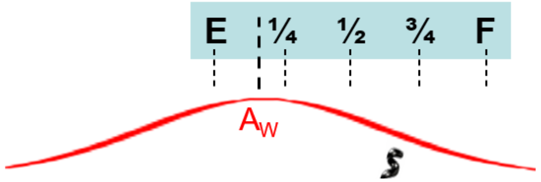
- **Mystery: why is the lowering operator appearing?**

Example: Gently measure X so that you don't disturb P

- What if we do a weak measurement of X, and then make a strong measurement of P?

i.e. **A** $= |x\rangle\langle x| = \pi$, Initial state= $|\psi\rangle$, Strong measurement result $P=p$

Average shift of the pointer: $A_W = \dfrac{\langle b|A|\psi\rangle}{\langle b|\psi\rangle}$



$$\pi_w = \frac{\langle p|x\rangle\langle x|\psi\rangle}{\langle p|\psi\rangle}$$

And if $p=0$, $$\pi_w = \frac{1/\sqrt{2\pi}\cdot\langle x|\psi\rangle}{\sqrt{\mathrm{Prob}(p=0)}} = \boxed{k\cdot\psi(x)}$$

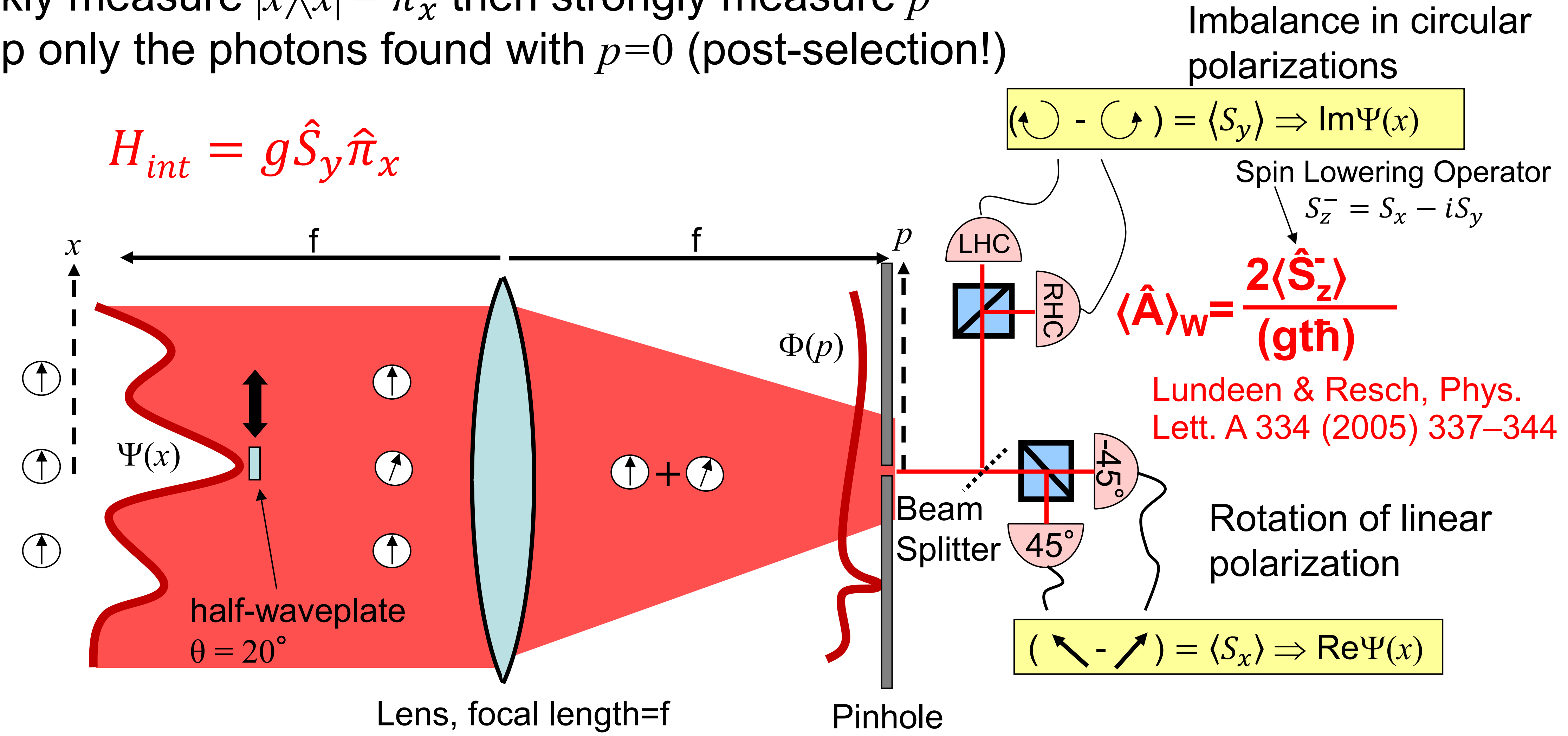
- **The average shift of the pointer (i.e. rotation of the polarization) is proportional to the wavefunction,**

Lundeen Nature, 474, 188 (2011)

Direct Measurement of the Wavefunction

- Weakly measure $|x\rangle\langle x| = \hat{\pi}_x$ then strongly measure p
- Keep only the photons found with p=0 (post-selection!)

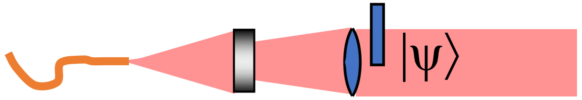
$$H_{int} = g\hat{S}_y\hat{\pi}_x$$



- The average result of the weak measurement is the real and imaginary components of the wavefunction

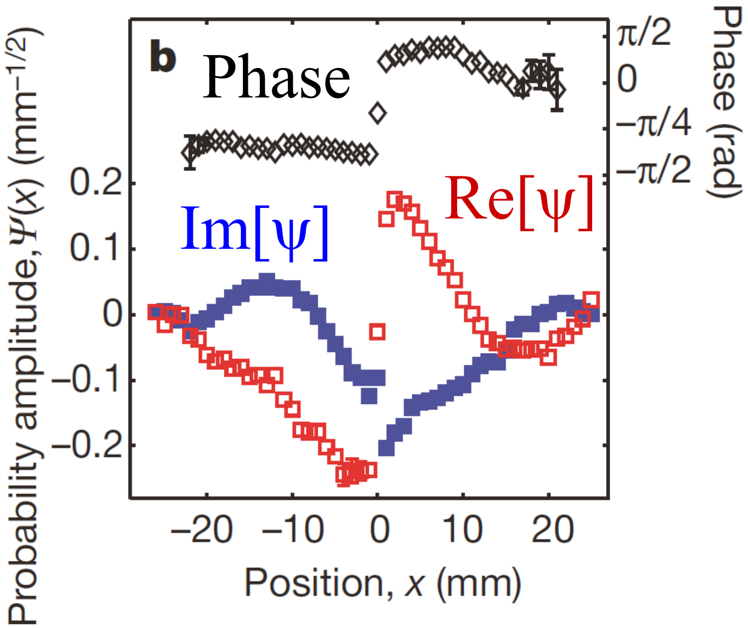
Direct Measurement of the Wavefunction

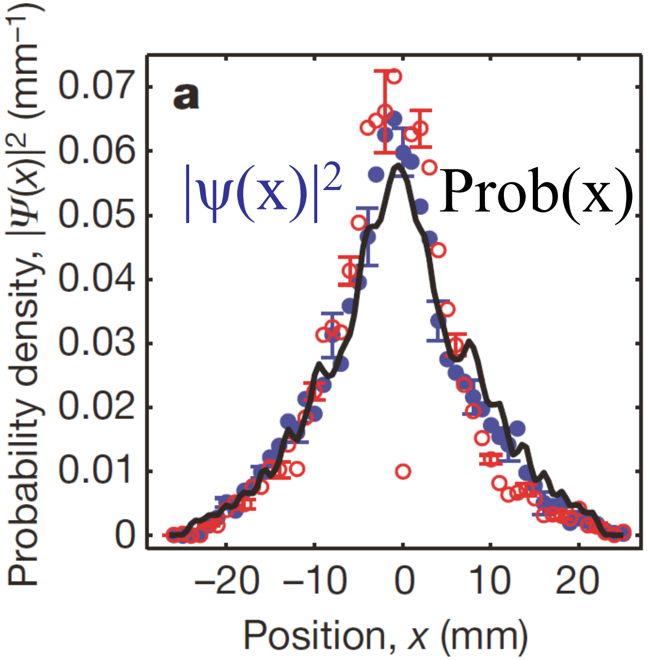
- Demonstrate method with $\Psi(x)$ of photons exiting a single-mode fibre



(↺ - ↻) ⇒ $\mathrm{Im}\Psi(x)$

(↖ - ↗) ⇒ $\mathrm{Re}\Psi(x)$

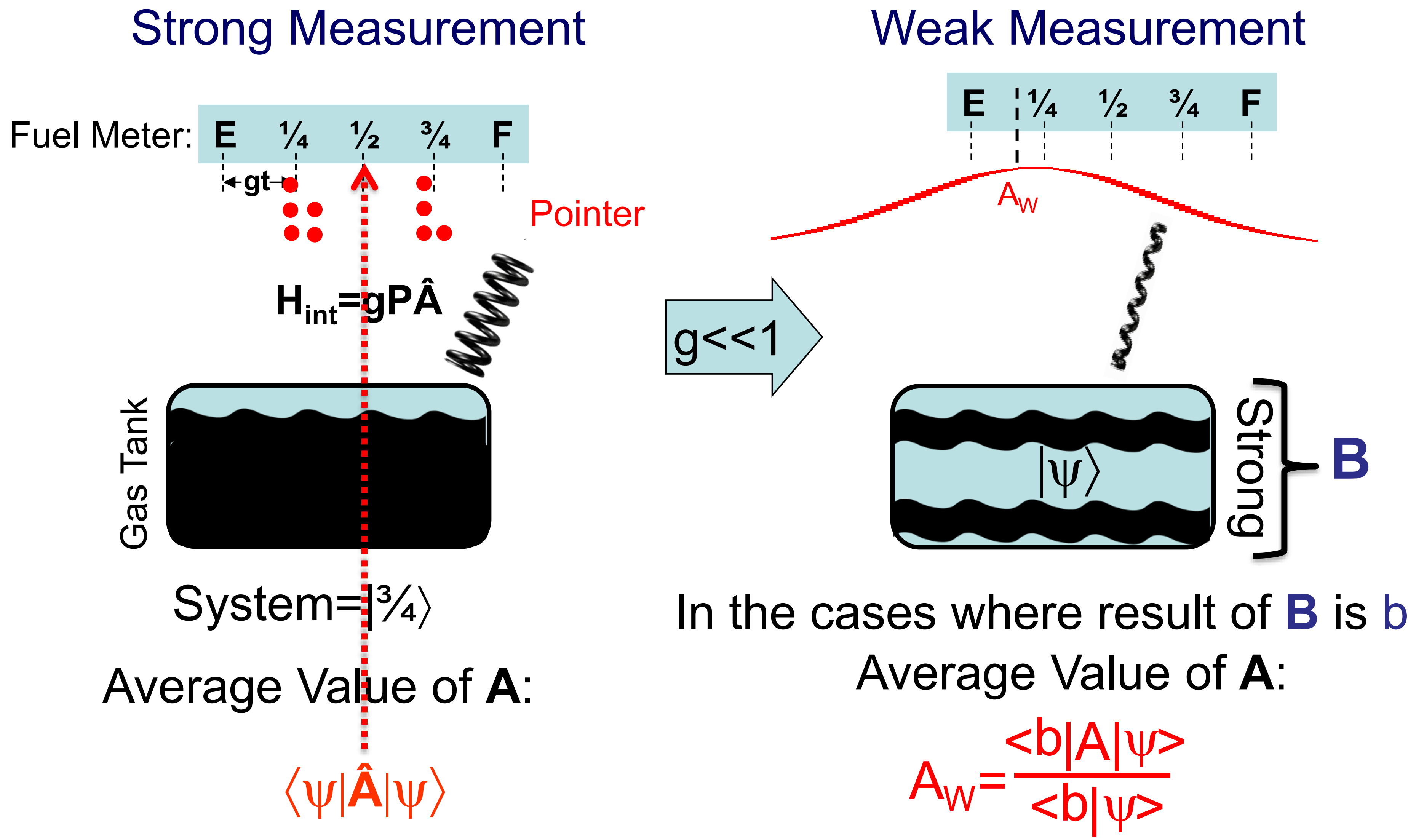




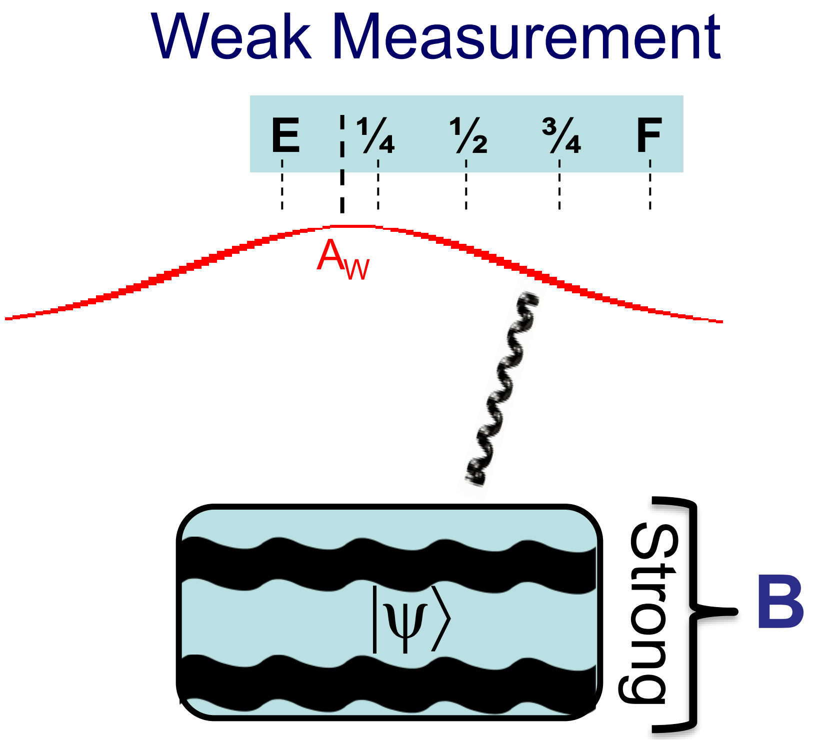
Lundeen Nature, 474, 188 (2011)

- The two signals directly give Im[ψ] and Re[ψ].
- Direct measurement accurately shows phase and magnitude of ψ(x)

Weak Measurement



Weak Measurement



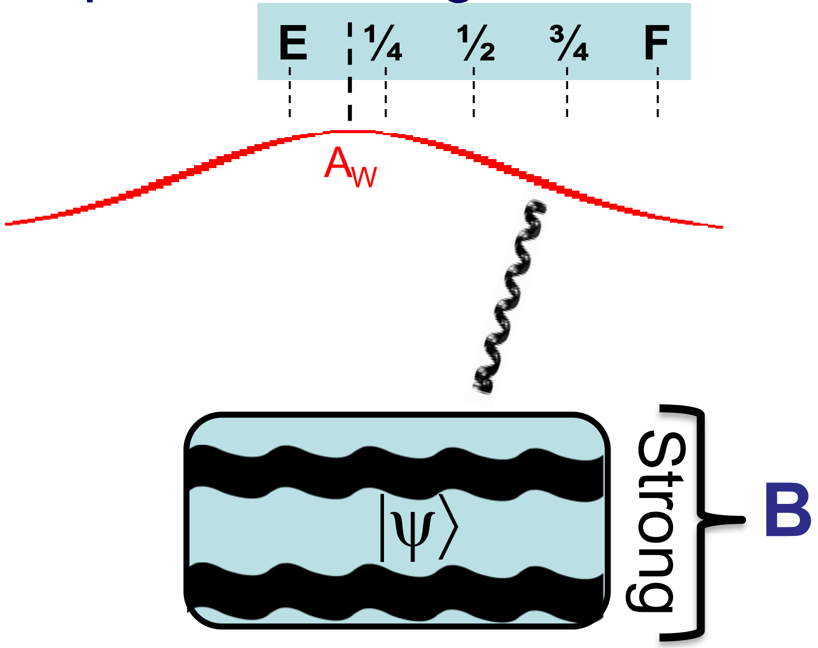
In the cases where result of **B** is b

Average Value of **A**:

$$A_W = \frac{\langle b|A|\psi\rangle}{\langle b|\psi\rangle}$$

Weak Measurement without post-selection

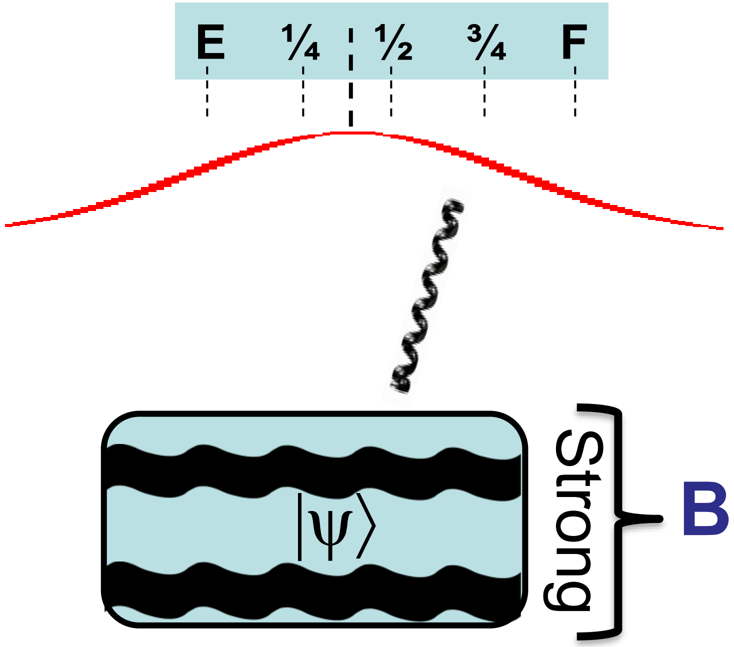
Weak measurement W of A,
post-selecting on b



In the cases where result of **B** is b
Average Value of **A**:

$$A_W = \frac{\langle b|A|\psi\rangle}{\langle b|\psi\rangle}$$

Weak Measurement W of A
and strong S of B



Average Value of **AB** ≡ *Weak Average:*

$$\langle A_W B_S\rangle = \langle\psi|AB|\psi\rangle = Tr[AB\rho]$$

- A weak-strong measurement of AB is just the regular quantum expectation value.
- If AB don't commute, $\langle A_W B_S\rangle = Tr[AB\rho]$ can be complex

Weak average and the lowering operator

The weak average is $\langle A_W B_S\rangle = \langle\psi|AB|\psi\rangle = \langle AB\rangle$

The weak average is found by

$$\langle A_W B_S\rangle = \frac{2\sigma}{gt}\left\langle a_{Pointer}\ B_{System}\right\rangle_{Pointer+system}$$

$$Re\langle A_W B_S\rangle \propto \left\langle x_{Pointer}\ B_{System}\right\rangle_{Pointer+system}$$

$$Im\langle A_W B_S\rangle \propto \left\langle p_{Pointer}\ B_{System}\right\rangle_{Pointer+system}$$

- The weak average is a correlation between the pointer and system

Aside: The weak value and weak average obey Baye's law

If A and B are projectors $\pi_{A=a}$ and $\pi_{B=b}$ then the expectation values are probabilities.

$Prob(b) = \langle \pi_b \rangle$ so…. $Prob(a, b) = \langle \pi_a \pi_b \rangle$, the weak average

So by Baye's law: $Prob(a|b) = \frac{Prob(a,b)}{Prob(b)} = \frac{\langle \psi | \pi_a \pi_b | \psi \rangle}{\langle \psi | \pi_b | \psi \rangle} = \frac{\langle b | \pi_a | \psi \rangle}{\langle b | \psi \rangle}$, the weak value.

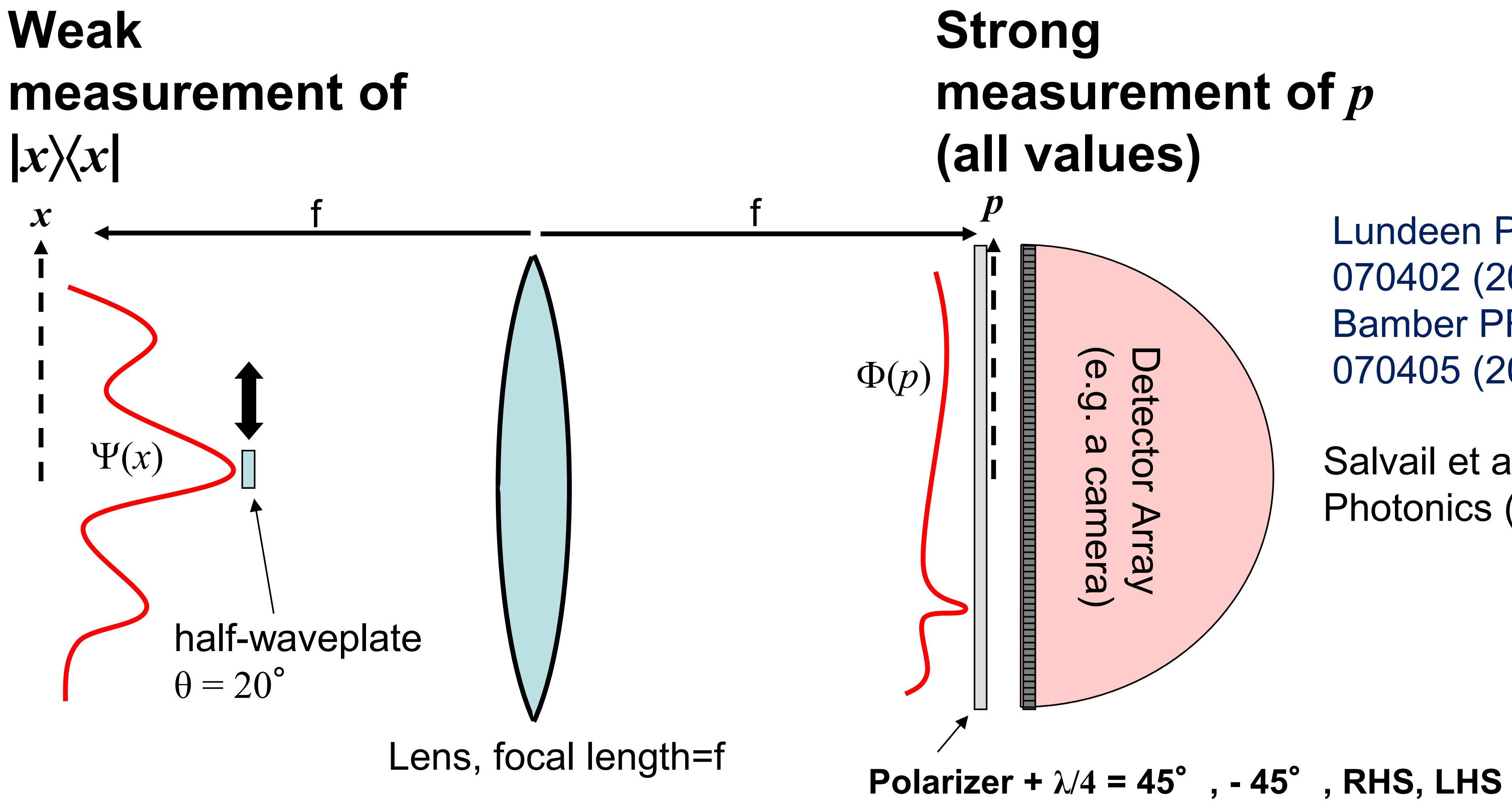
Steinberg, A. M., *Phys. Rev. A* 52, 32 (1995).

H. F. Hofmann, New Journal of Physics, 14, 043031 (2012).

If the weak average $\langle \pi_a \pi_b \rangle_W = Prob(a, b)$
then the weak value $\langle \pi_a \rangle_{W,b} = Prob(a|b)$

- The weak value + weak average obey Baye's law despite being complex quantities.
- The strange formula for the weak value $\Leftrightarrow$ conditional probability.

No post-selection: Joint measurement of X and every P



Lundeen PRL 108, 070402 (2012), Bamber PRL 112, 070405 (2014)

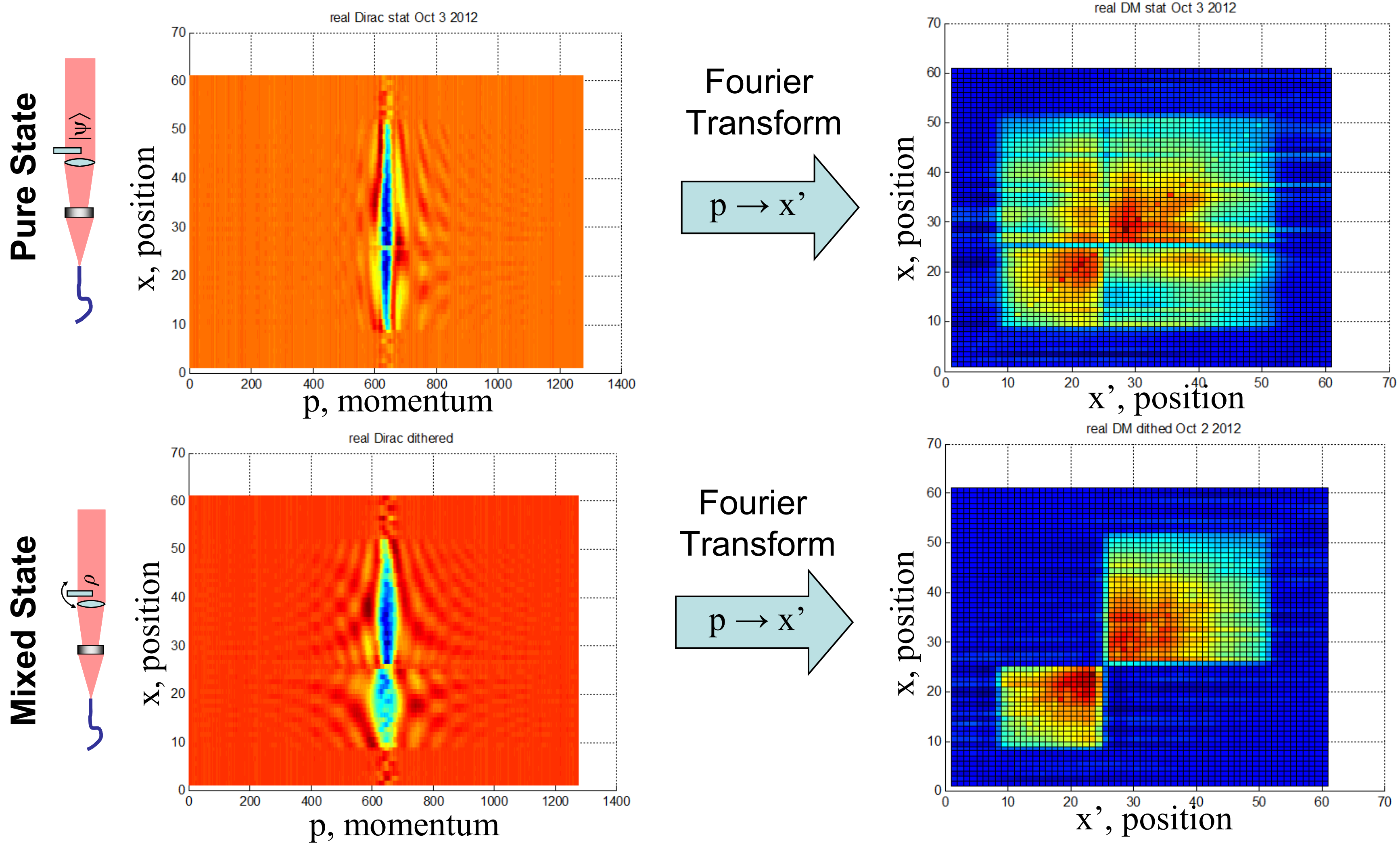
Salvail et al. Nature Photonics (2013)

• Joint measurement of $\pi_x = |x\rangle\langle x|$ and $\pi_p = |p\rangle\langle p|$ gives the Kirkwood-Dirac Distribution:

$$KD(x,p) = \langle \pi_x \, \pi_p \rangle = \mathrm{Tr}[\pi_x \, \pi_p \, \rho]$$

Direct measurement of the Kirkwood - Dirac Distribution

Kirkwood-Dirac Distribution, $KD(x,p) = \mathrm{Tr}[\pi_x \pi_p \rho]$ Density Matrix $\rho(x,x')$



- Simple generalization allows us to completely measure mixed states

Lundeen PRL 108, 070402 (2012), Bamber PRL 112, 070405 (2014),

Direct Measurement of an Entangled Quantum State

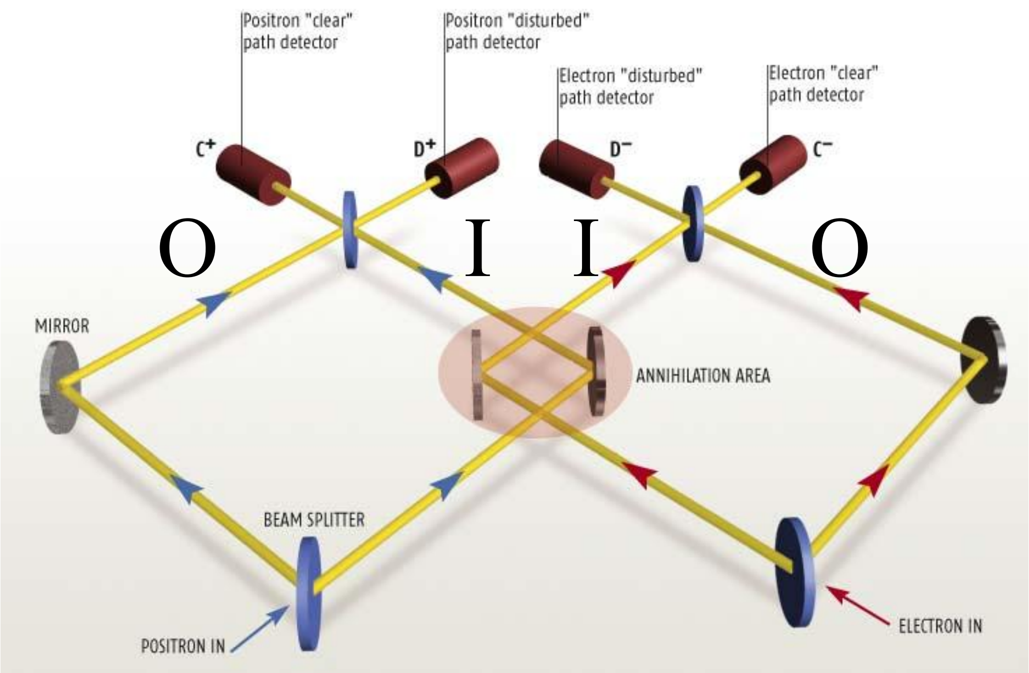
PRL 102, 020404 (2009) PHYSICAL REVIEW LETTERS week ending 16 JANUARY 2009

Experimental Joint Weak Measurement on a Photon Pair as a Probe of Hardy's Paradox

J. S. Lundeen and A. M. Steinberg

HARDY'S PARADOX

The positron and electron go down both arms of each of their interferometers. If they meet in the overlapping arms, they should annihilate each other. But, bizarrely, they are still registered as arriving at the D detectors



Theoretical Quantum State:

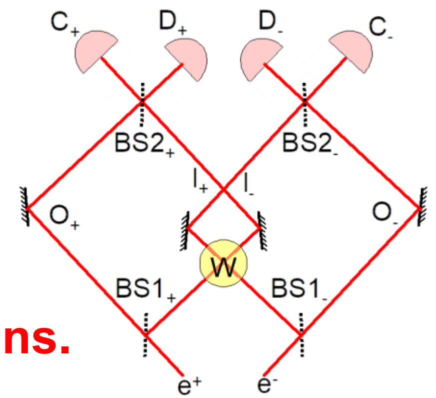
$$|\psi\rangle = 1 \quad |IO\rangle \quad +1 \quad |OI\rangle \quad -1 \quad |OO\rangle \quad +0 \quad |II\rangle$$

Two-Particle Weak Measurements

$$|\psi\rangle = \mathbf{1}\ |IO\rangle + \mathbf{1}\ |OI\rangle\ \mathbf{-1}\ |OO\rangle + \mathbf{0}\ |II\rangle$$

Problem: To directly measure this we need to measure two-particle observables

e.g., $|IO\rangle\langle IO| = \hat{A}_1\hat{A}_2 = |I\rangle\langle I|_1\ |O\rangle\langle O|_2$



For two-particle weak measurements we need a strong optical nonlinearity to implement a Von Neuman measurement interaction ($H_{int}=gP\hat{A}_1\hat{A}_2$).

Solution: **Conduct two single-particle weak measurements and study pointer correlations.**

$$H_{int} = g\hat{S}_{1y}\hat{A}_1 + g\hat{S}_{2y}\hat{A}_2$$

Two-particle strong measurement with spin pointers:	Two-particle weak measurement with spin pointers and post-selection
$\langle \hat{A}_1\hat{A}_2 \rangle = \dfrac{\langle \hat{S}_{1x}\hat{S}_{2x} \rangle}{(gt\hbar s)^2}$	Spin Lowering Operators $\langle \hat{A}_1\hat{A}_2 \rangle_W = \dfrac{\langle \hat{S}_1^-\hat{S}_2^- \rangle_{fi}}{(gt\hbar s)^2}$ Lundeen & Resch, Phys. Lett. A 334 (2005) 337–344 Resch & Steinberg, PRL 92,130402 (2004)

Directly Measuring Entangled States

- Weakly measure where the particle pair is in Hardy's Paradox, i.e., in the su... ...le of photon pairs that exit a the Dark (D) detectors (post-selection on com... ...vable)

measure e.g. $\pi_{IO} = |\text{Inner Outer}\rangle\langle\text{Inner Outer}| = ...$

Direct measurement of the Quantum State:

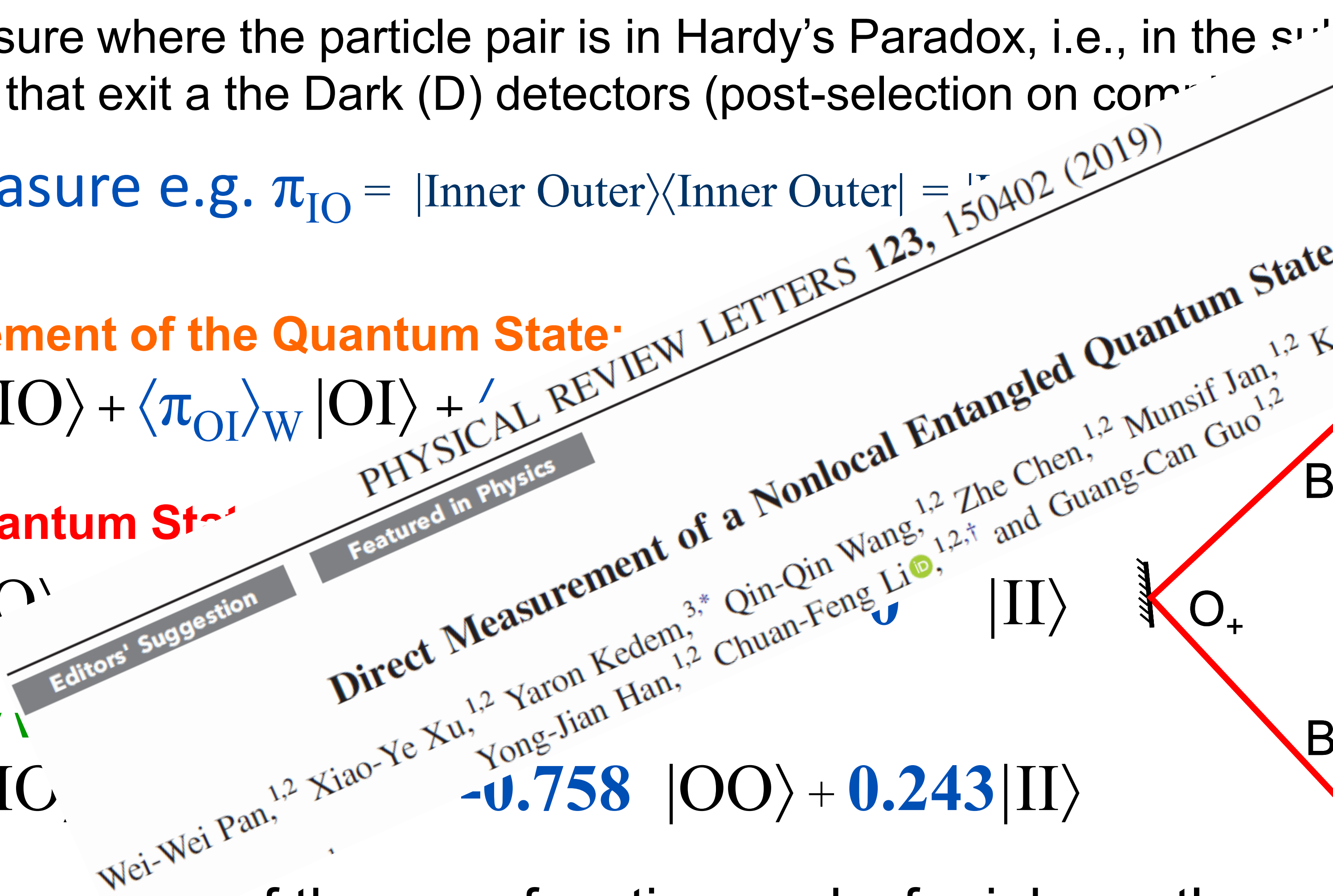
$$|\psi\rangle = \langle\pi_{IO}\rangle_W |IO\rangle + \langle\pi_{OI}\rangle_W |OI\rangle + ...$$

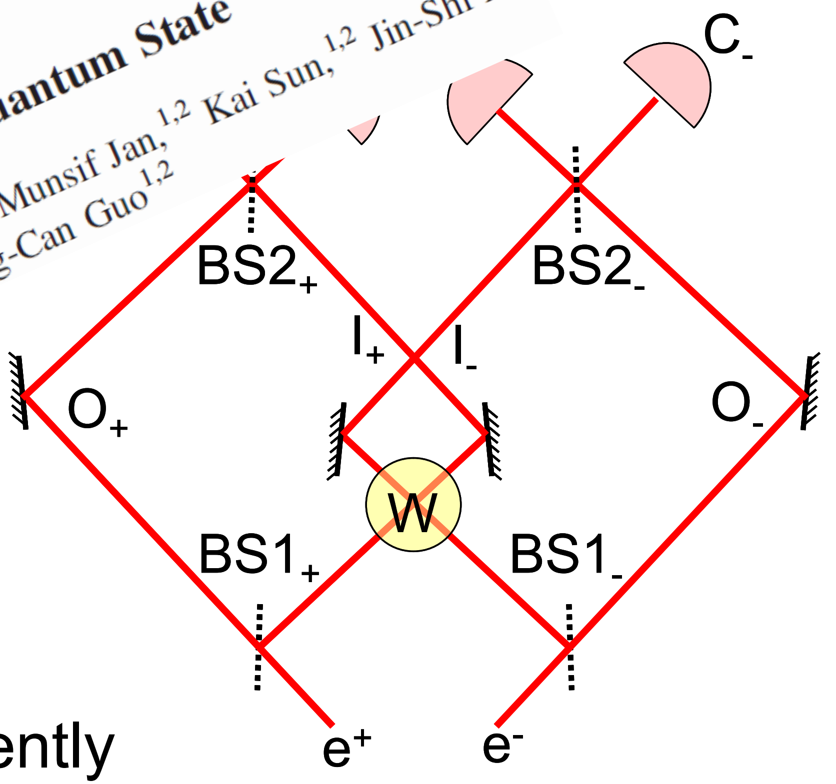
Theoretical Quantum St...

$$|\psi\rangle = \mathbf{1} \quad |IO\rangle ... \quad |II\rangle$$

Experimentally ...

$$|\psi\rangle = \mathbf{0.663}\, |IO\rangle ... \mathbf{-0.758}\, |OO\rangle + \mathbf{0.243} |II\rangle$$





- Direct measure... of the wavefunction works for inherently quantum systems (i.e. entangled particles).

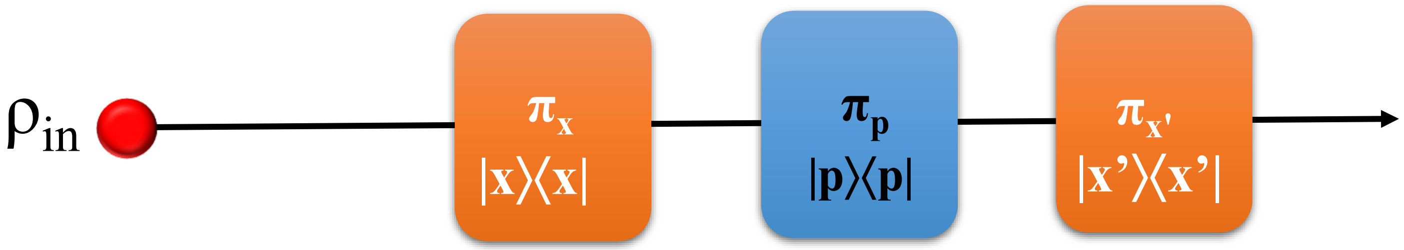
PRL 102, 020404 (2009) PHYSICAL REVIEW LETTERS week ending 16 JANUARY 2009

Experimental Joint Weak Measurement on a Photon Pair as a Probe of Hardy's Paradox

J. S. Lundeen and A. M. Steinberg

Directly Measuring the Density Matrix

- Jointly weakly measure X then P then X again



Theory: Lundeen & Bamber PRL 108, 070402 (2012).

Average result is $\mathrm{Tr}[\pi_x \, \pi_p \, \pi_{x'} \, \rho_{in}] = \rho_{in}(x,x')$

How can we measure this?

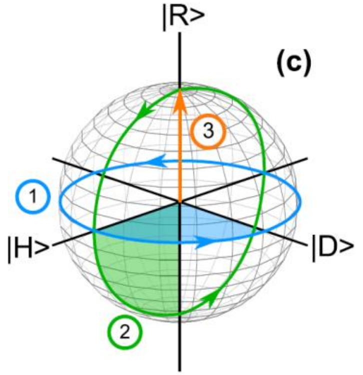
Step 1: $H_{int} = gP_1\pi_x$ using pointer 1
Step 2: $H_{int} = gP_2\pi_p$ using pointer 2
Step 3: strongly measure $\pi_{x'}$
Step 4: $\langle a_1 a_2 \pi_{x'} \rangle_{system+pointer} \propto Tr[\pi_x \, \pi_p \, \pi_{x'} \, \rho_{in}]$

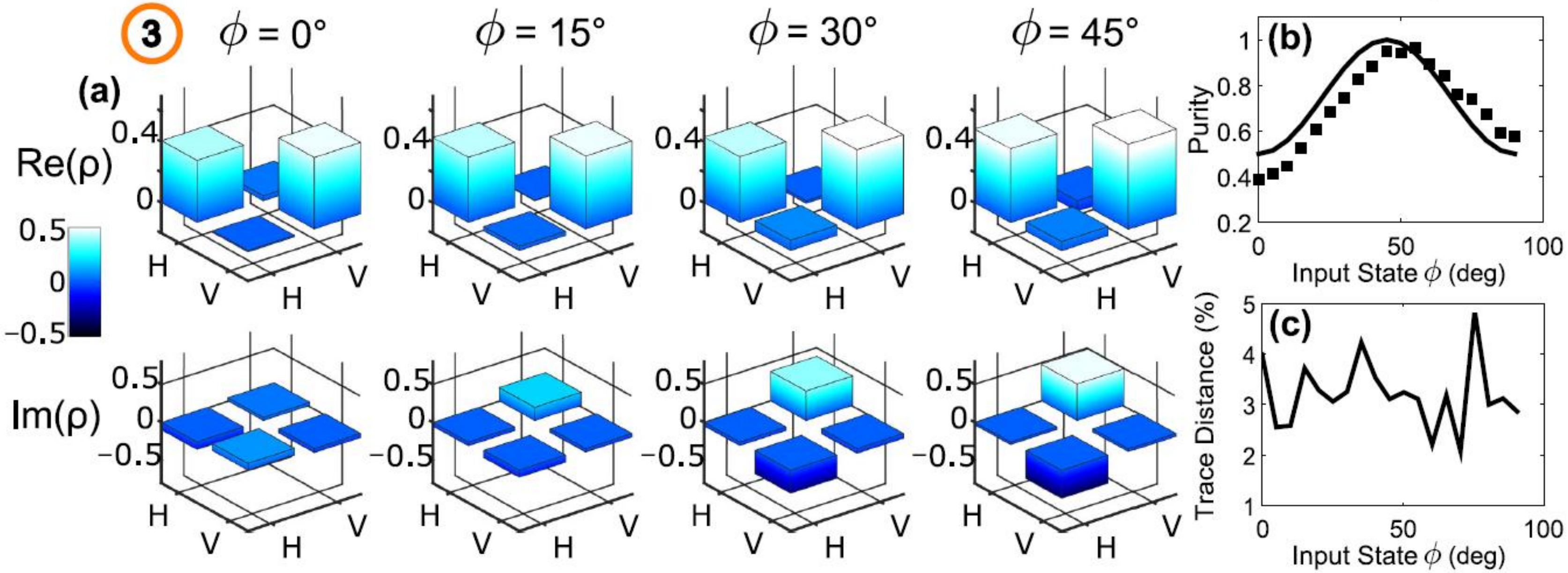
- We can know any chosen element $\rho_{in}(x,x')$ of the density matrix e.g. a particular coherence, entanglement witnesses, etc.

Direct Measurement of the Density Matrix

Thekkadath PRL 117, 120401 (2016)

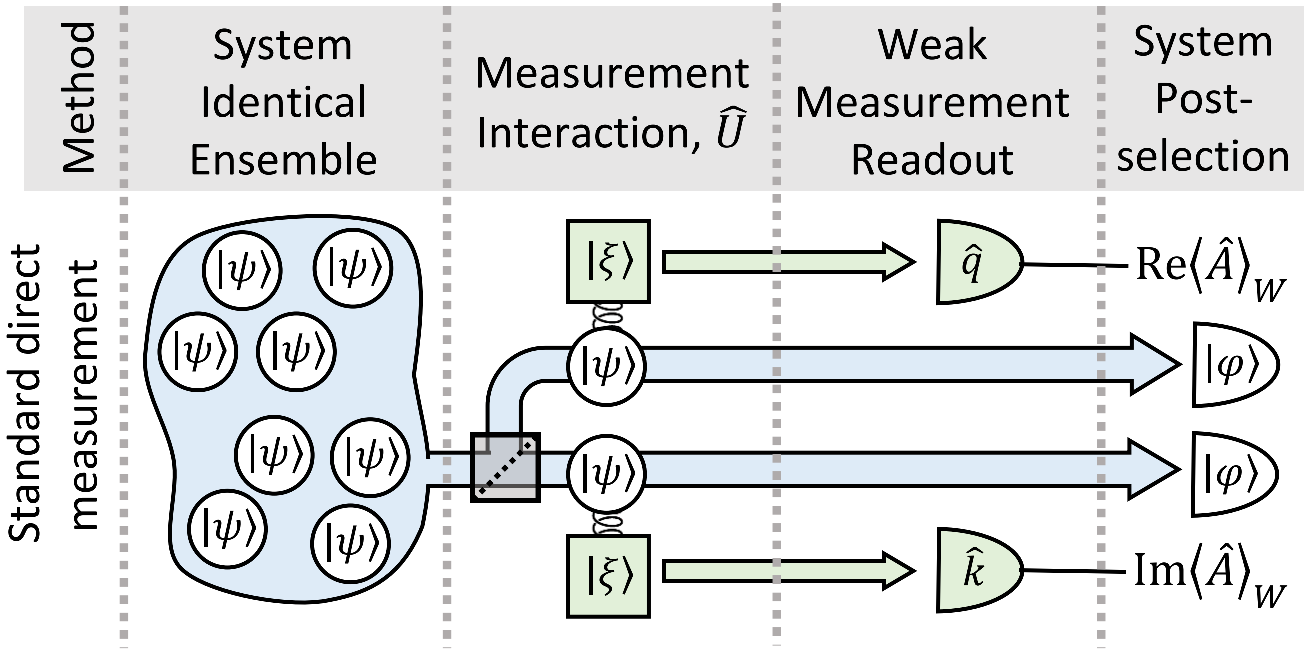
We measured the density matrix of a polarized photon
Testing states along path 3 in the Poincaré sphere





- The measured and expected purities match
- The trace distance between the expected and measured density matrices is less than 0.05

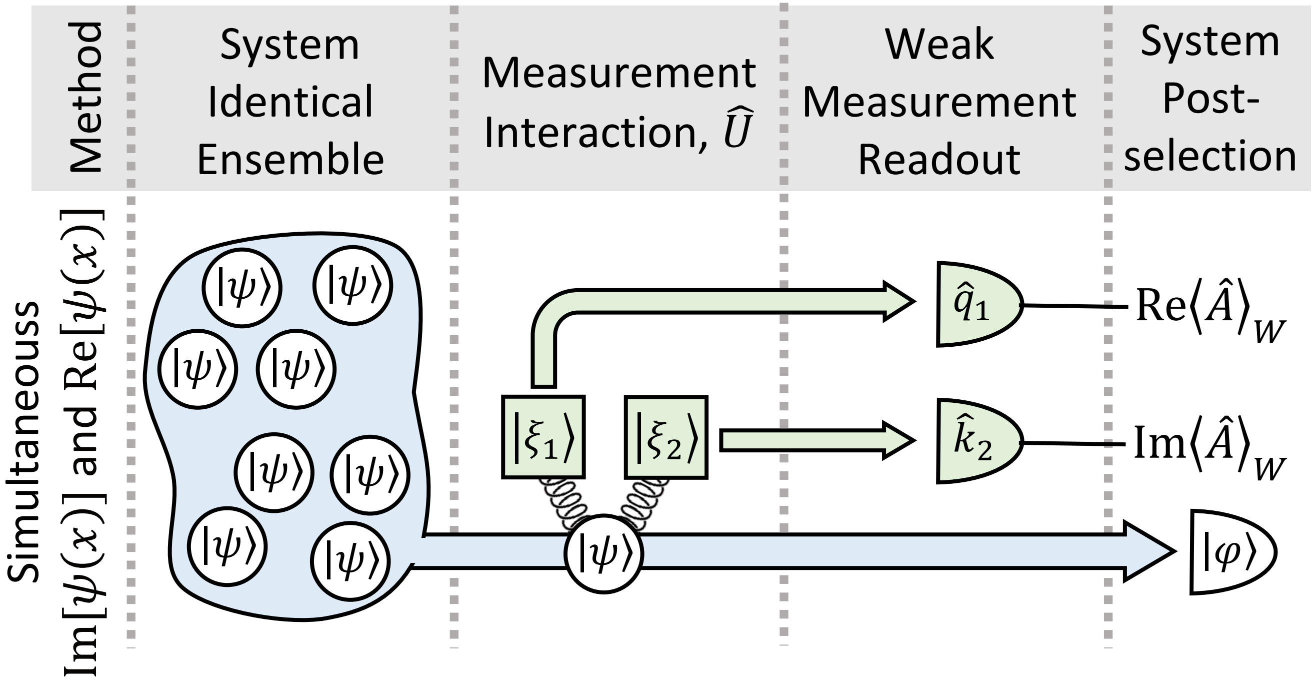
Even more direct?



- We switch back and forth between measuring $\mathrm{Im}[\psi(x)]$ and $\mathrm{Re}[\psi(x)]$
- Can we measure both in each trial?

Even more direct: Simultaneous readout

- Solution: Weak measurements do not disturb each other
  ∴ Weakly measure twice in row, once for $\mathrm{Im}[\psi(x)]$ and once $\mathrm{Re}[\psi(x)]$
- Need two readouts (i.e. 'pointers') or a two-dimensional readout.



PHYSICAL REVIEW A **100**, 032119 (2019)

Experimental simultaneous readout of the real and imaginary parts of the weak value

A. Hariri, D. Curic, L. Giner, and J. S. Lundeen*

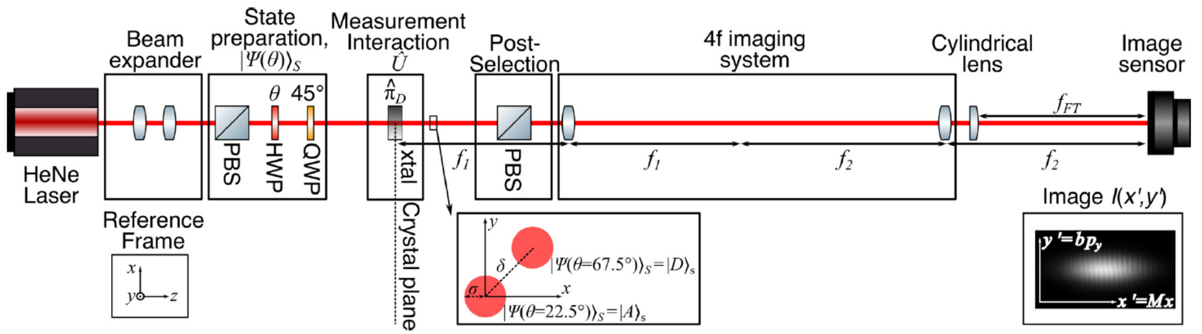
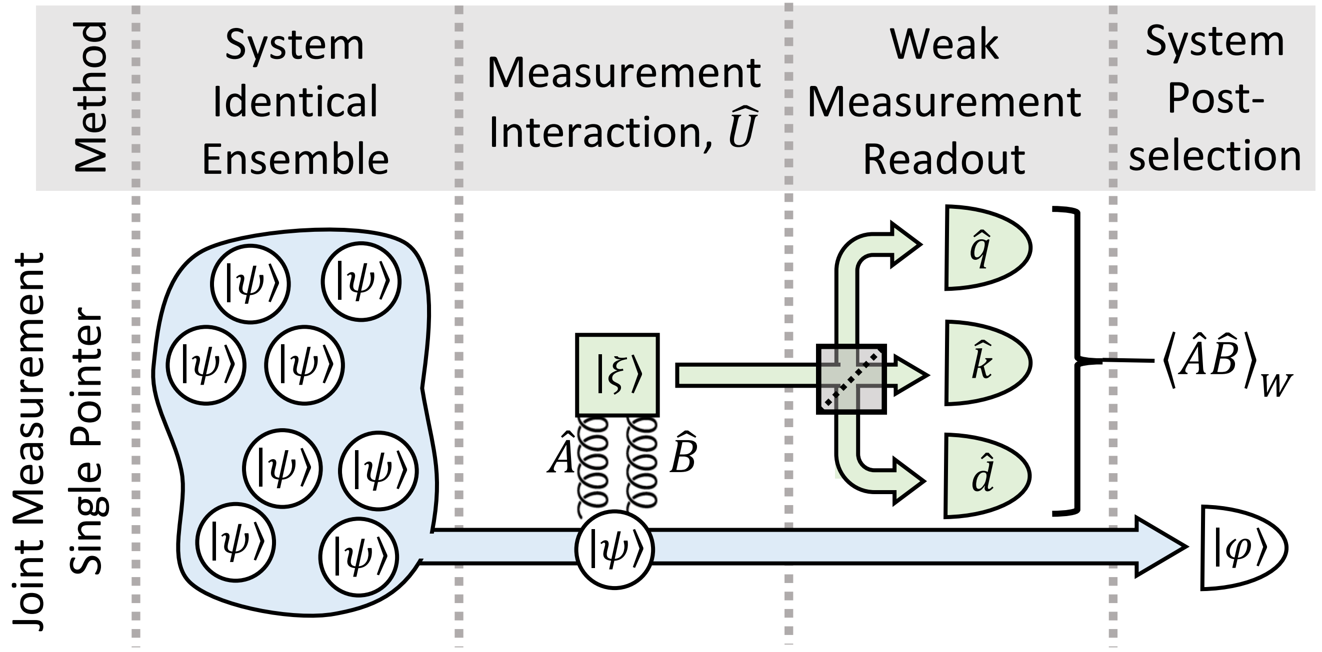
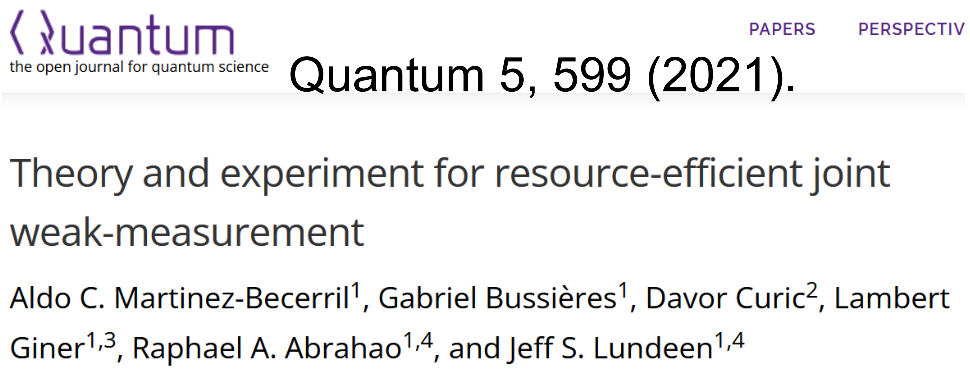


FIG. 2. Experimental setup for simultaneous readout of the real and imaginary parts of the weak value. This setup implements method C in

Better Joint Measurements of AB

- Needed one readout system ('pointer') per observable
- Here, only need a single readout system for multiple projectors, $|a\rangle\langle a|, |b\rangle\langle b|$
- But, need to measure more readout system observables

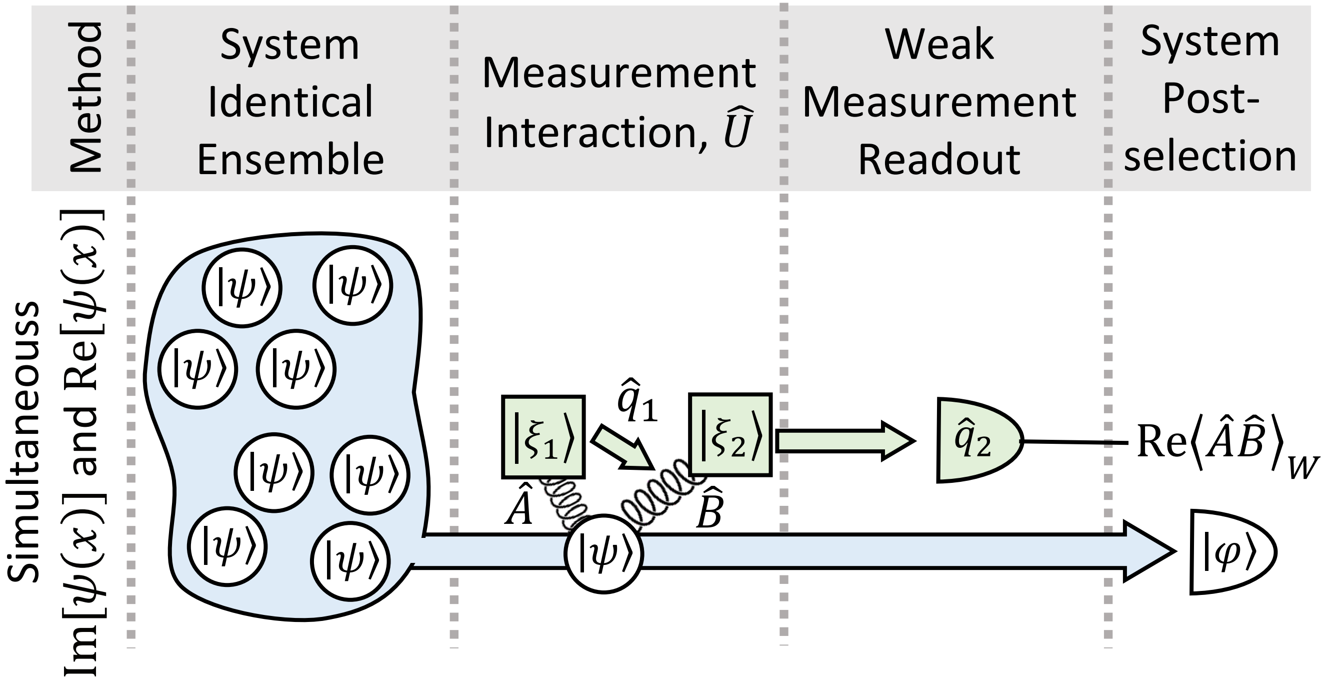




Quantum 5, 599 (2021).

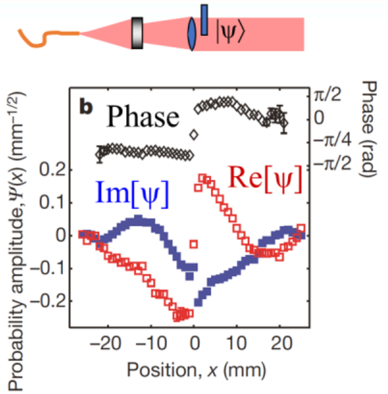
Even more direct: Simultaneous readout

- Want simple (i.e. 'direct') readout of a single pointer system
- Solution: Measure B scaled by the outcome of the measurement of A
- Condition the strength of the measurement of B on the outcome of A.

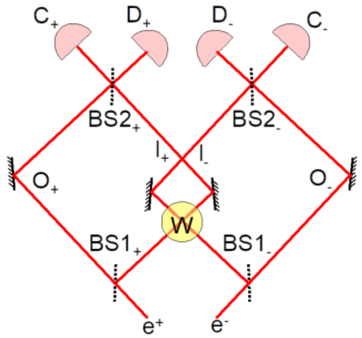


Theory: Lundeen, Bamber, PRL 108, 070402 (2012)
Previous talk by Michael Weil

Conclusions

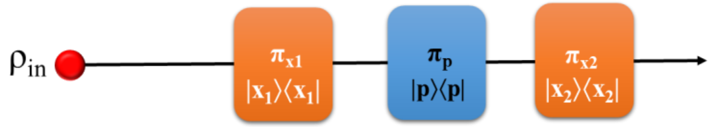


1. Measurements of complementary variables by weak measurement give the quantum state of a system, wave-function or Kirkwood-Dirac



2. Generalization of two-particle correlation measurements to weak von-Neumann interactions allows us to measure entangled states

3. Generalization of three-time correlation measurements to weak von Neumann interactions allows us to measure the density matrix



Wavefunction: Lundeen Nature, 474, 188 (2011)
Mixed States: Lundeen PRL 108, 070402 (2012), Bamber PRL 112, 070405 (2014),Thekkadath PRL 117, 120401 (2016), Thekkadath PRL 119, 050405 (2017)
Heisenberg: Thekkadath NJP 20, 113034 (2018)

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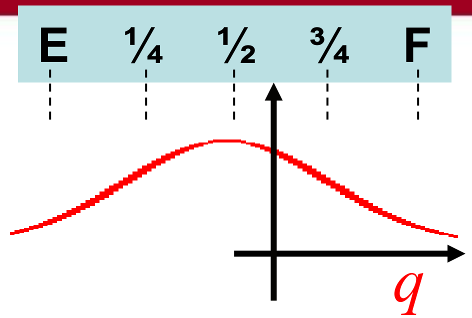
Uncertainty and Weak Measurement



- The weak measurement POVM Π is a projector, $|\pi(q)\rangle$.
  - ➢ Superposition of sharp states in x and p

Measured pointer position, q

Weak measurement of $|x'\rangle\langle x'|$

Strong measurement of $|p'\rangle\langle p'|$

$$|\pi(q)\rangle = |x'\rangle + \mathcal{P}(q)e^{ix'p'}\,|p'\rangle$$

Predictability $\mathcal{P}(q)$ is our ability to predict whether the particle had x' given outcome q.

- In the double-slit experiment, predictability $\mathcal{P}$ and visibility $\mathcal{V}$ obey an uncertainty relation:

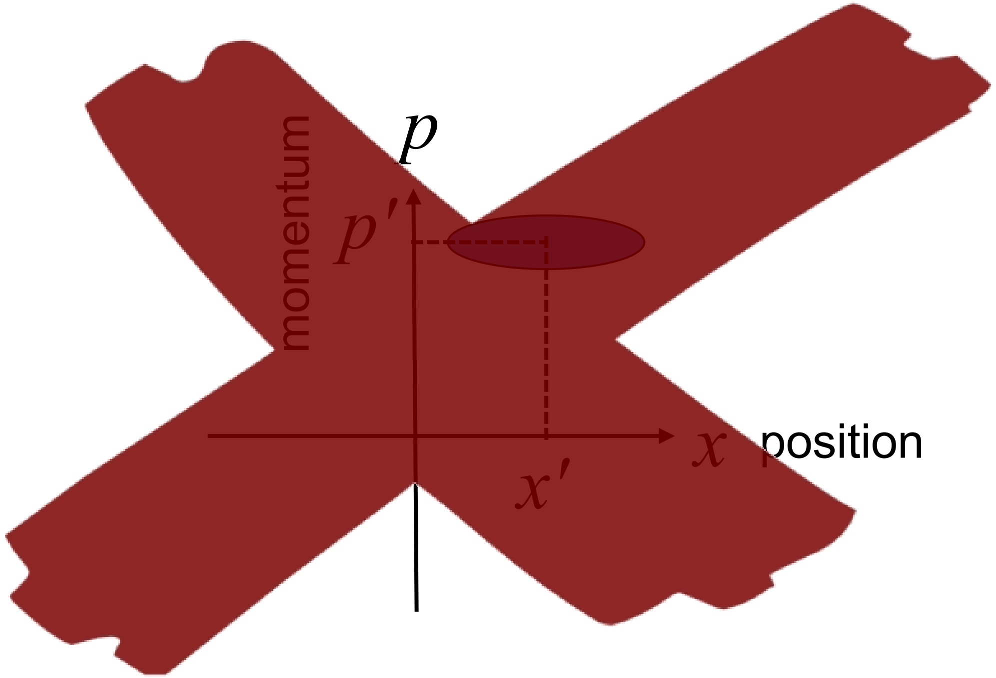
$$\mathcal{P}^2 + \mathcal{V}^2 \ \leq 1$$

- Weak measurement trades away predictability to reduce disturbance to the quantum coherence (i.e. visibility)

GS Thekkadath, F Hufnagel, JS Lundeen, New J Phys 20, 113034 (2018)

Compatibility with the Heisenberg Uncertainty Relation

- Weak measurements reduce disturbance at the expense of certainty.
- Do they trade precision in Δp for imprecision in Δx?
- What does the POVM Π of the measurement look like in phase-space?



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