Max Planck - University of Ottawa Centre
for Extreme and Quantum Photonics

Lundeen Lab
Quantum Photonics

Chaining weak measurements:
A direct approach to joint measurements of position and momentum

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Motivation

First: Incompatible observables correspond to non-commuting operators: $[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A} \neq 0$

"Physical" consequences:

1. These observables cannot be measured sequentially since a measurement of one disturbs the other:

$$\hat{A}\hat{B}|\psi\rangle \neq \hat{B}\hat{A}\,|\psi\rangle$$

2. There is an intrinsic limit to the precision with which $\hat{A}$ and $\hat{B}$ can be simultaneously determined:

$$\Delta\hat{A}\Delta\hat{B} \geq \frac{1}{2}\left|\langle[\hat{A}, \hat{B}]\rangle\right|$$

However, it *is* possible to "simultaneously" measure the two observables to find their joint value $\langle\hat{A}\hat{B}\rangle$

"Joint measurements" $\langle\widehat{\mathbf{A}}\widehat{\mathbf{B}}\rangle$

Joint values

Joint values can be used to:

1. Investigate fundamental quantum dynamics
2. Investigate entanglement
3. Reconstruct quantum states

Joint measurements have been made:

1. Using optimally cloned states [1,2]
2. Using fractional Fourier transforms [3]

Unfortunately, these schemes all require post-processing of measurements and/or the readout of the states of multiple pointers.

1. G M D'Ariano, C Macchiavello, and M F Sacchi., J. Opt. B: Quantum Semiclass. Opt. 3.2 (2001)
2. G. S. Thekkadath et al., Phys. Rev. Lett. 119 (2017)
3. Aldo C Martinez-Becerril et al., Quantum 5 (2021)

Using a **conditional** measurement scheme allows the joint operator's **weak** average to be directly inferred from a single pointer's position [4].

Goal: measure $\langle \hat{\pi}_p \hat{\pi}_x \rangle$ of a *photon*

The Kirkwood-Dirac (KD) distribution is a quasi-probability distribution whose elements depend on the joint value of the position and momentum projector operators [4,5]:

$\Rightarrow$ Informationally equivalent to the density matrix

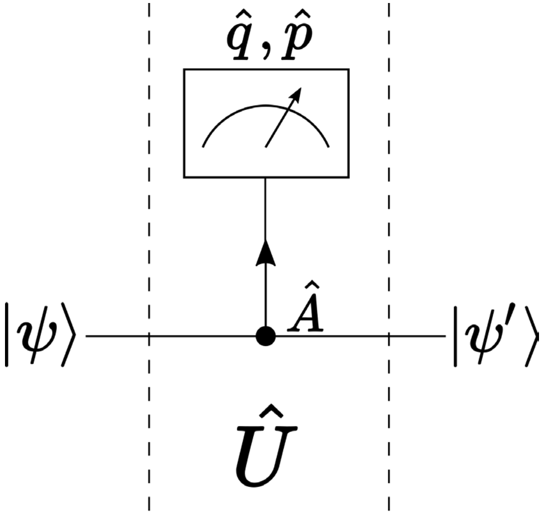
$\therefore$ Use conditional measurement of joint weak average for quantum state determination from single pointer readout

4. Jeff S. Lundeen and Charles Bamber, Phys. Rev. Lett. 108 (2012)
5. David Roland Miran Arvidsson-Shukur et al., New Journal of Physics 26 (2024)

Measurement

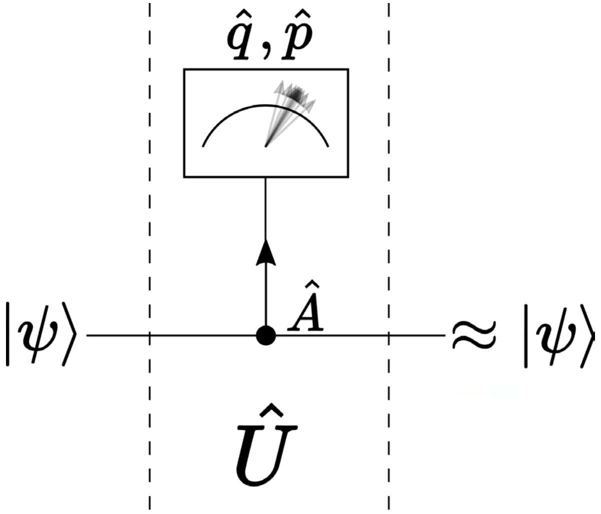
Standard von Neumann Measurement

- Measurement apparatus has a pointer that itself is described as a quantum state
- Pointer is coupled to the system being measured so that the deflection $\langle \hat{q} \rangle$ of the pointer reveals the value of the measured observable:



"Weak" Measurement

- Occurs when the coupling strength g of the interaction is small
- Measured system is minimally disturbed
- Much less information "shot-by-shot" but average still corresponds to the observable



$$\hat{U} = \exp\left(-ig\hat{A}\hat{p}/\hbar\right)$$

$$\rightarrow \langle \hat{q} \rangle \propto \langle \hat{A} \rangle$$

g: interaction strength
$\hat{A}$: observable of system
$\hat{p}$: conjugate momenta of pointer position

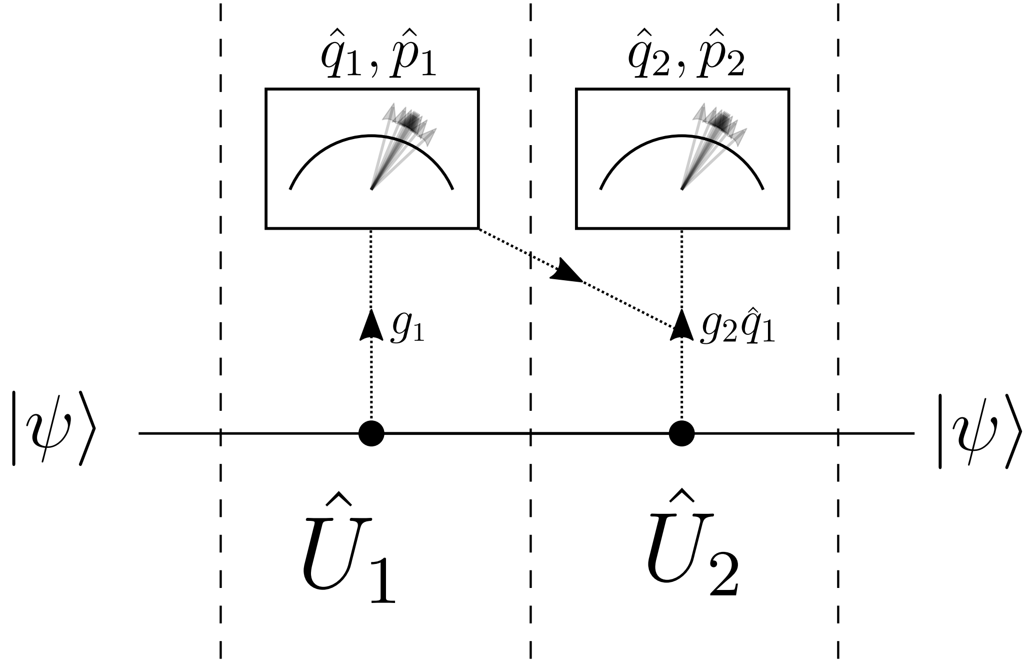
Chained weak measurements

Strength of a second measurement is **conditional** on the result of a first:

$$\hat{U} = \overbrace{\exp\left(-i(g_2\hat{q}_1)\hat{A}\hat{p}_2/\hbar\right)}^{\hat{U}_2} \times \overbrace{\exp\left(-ig_1\hat{B}\hat{p}_1/\hbar\right)}^{\hat{U}_1}$$

Allows the joint value to be directly read from the second pointer by itself:

$$\rightarrow \langle\hat{q}_2\rangle = g_1 g_2 \mathrm{Re}\left[\langle\psi|\hat{A}\hat{B}|\psi\rangle\right]$$



Procedure for Direct Measurement of General Quantum States Using Weak Measurement

Jeff S. Lundeen and Charles Bamber
Phys. Rev. Lett. **108**, 070402 – Published 13 February 2012

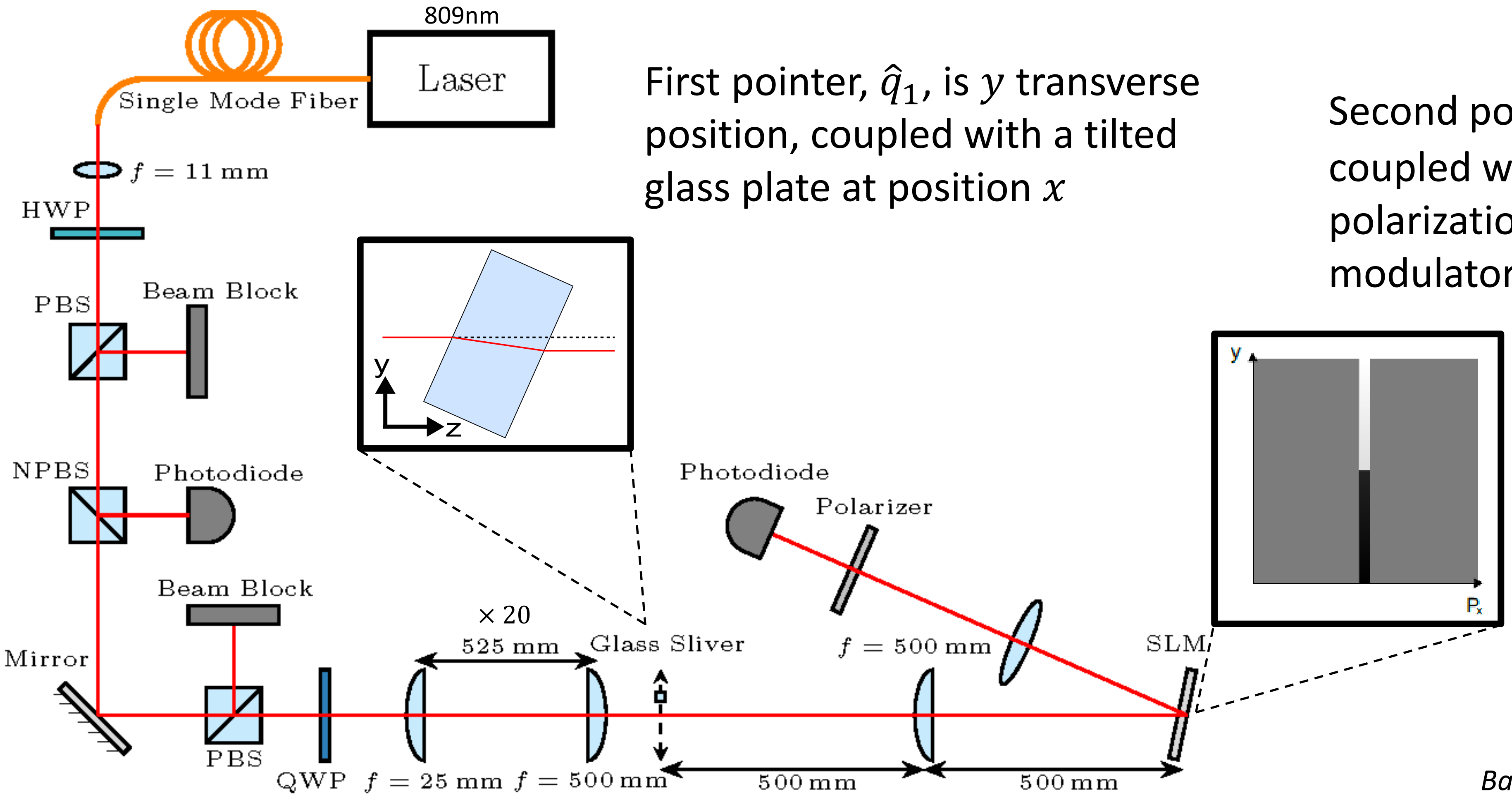
All the information about $\langle\hat{A}\hat{B}\rangle$ is now encoded in the position of the second pointer. This allows a subsequent strong measurement to "overwrite" the first pointer.

$$\hat{U} = \exp\left(-ig_3\hat{C}\hat{p}_1/\hbar\right) \times \exp\left(-ig_2\hat{q}_1\hat{A}\hat{p}_2/\hbar\right) \times \exp\left(-ig_1\hat{B}\hat{p}_1/\hbar\right)$$

$$\rightarrow \langle\hat{q}_1\rangle \propto \langle\hat{C}\rangle, \langle\hat{q}_2\rangle \propto \mathrm{Re}\left[\langle\hat{A}\hat{B}\rangle\right]$$

Experimental implementation

Observables are x transverse position, $\hat{A} = \hat{\pi}_x = |x\rangle\langle x|$, and p_x transverse momentum, $\hat{B} = \hat{\pi}_p = |p\rangle\langle p|$, of a *photon*.



First pointer, $\hat{q}_1$, is y transverse position, coupled with a tilted glass plate at position x

Second pointer, $\hat{q}_2$, is polarization ($\hat{s}_y$), coupled with a position dependent polarization rotation (using a spatial light modulator)

Note: setup can be scaled-down to a single-photon scheme.
E.g., using SPDC from 405nm light.

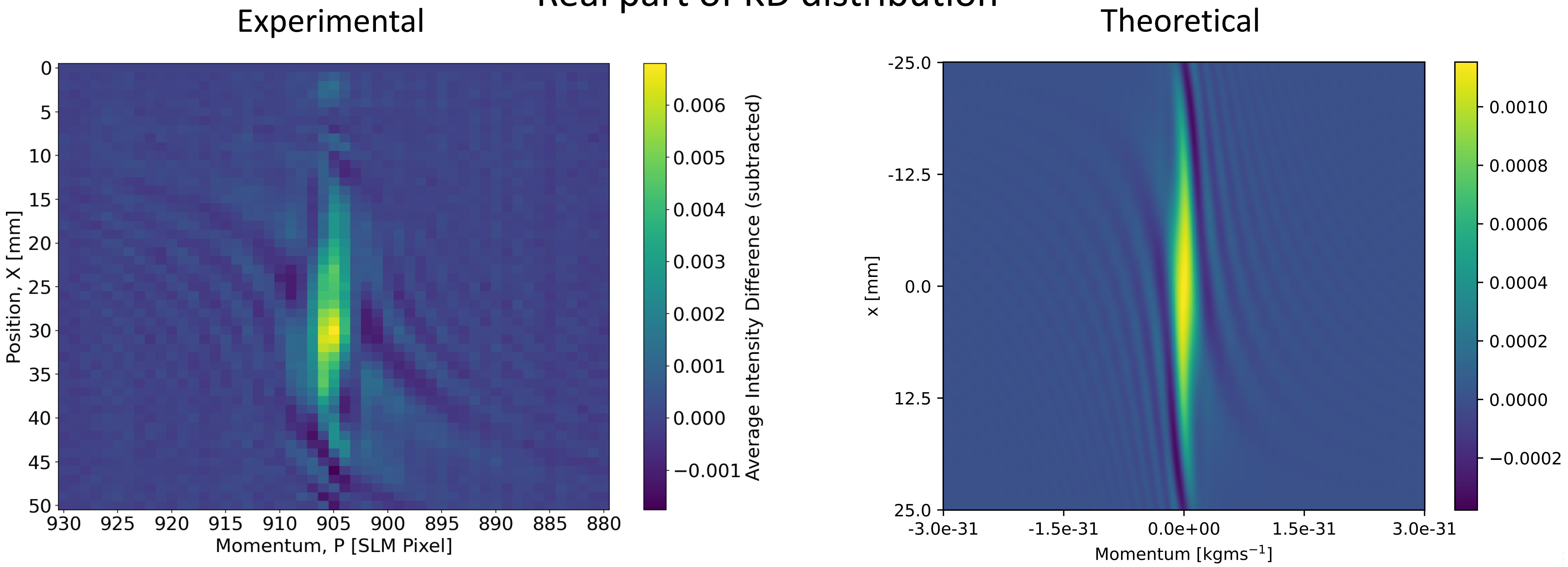
Bailey, T., Weil, M., Lundeen, J., To be submitted.

Results

Truncated Gaussian beam

Marginals correspond to the (Born rule) probability distributions expected for $\hat{\pi}_p$ and $\hat{\pi}_x$

Real part of KD distribution



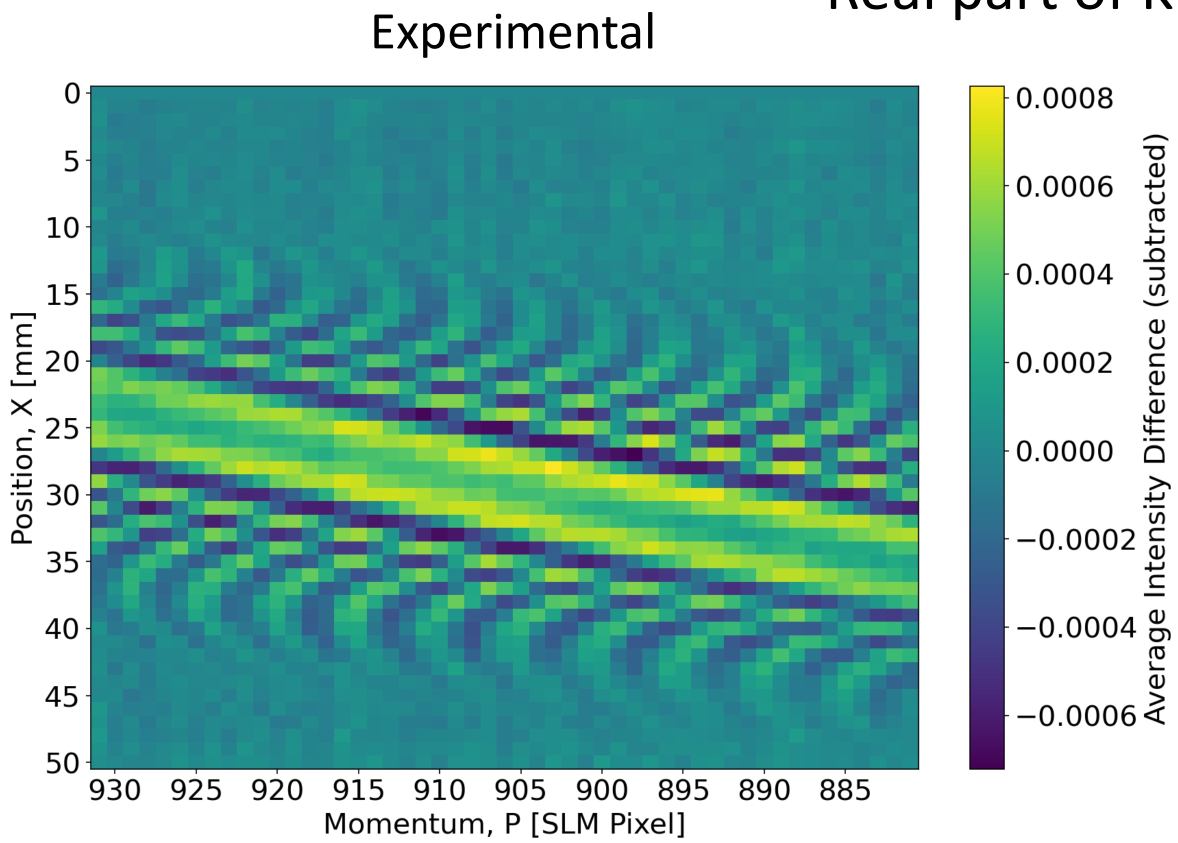
Bailey, T., Weil, M., Lundeen, J., To be submitted.

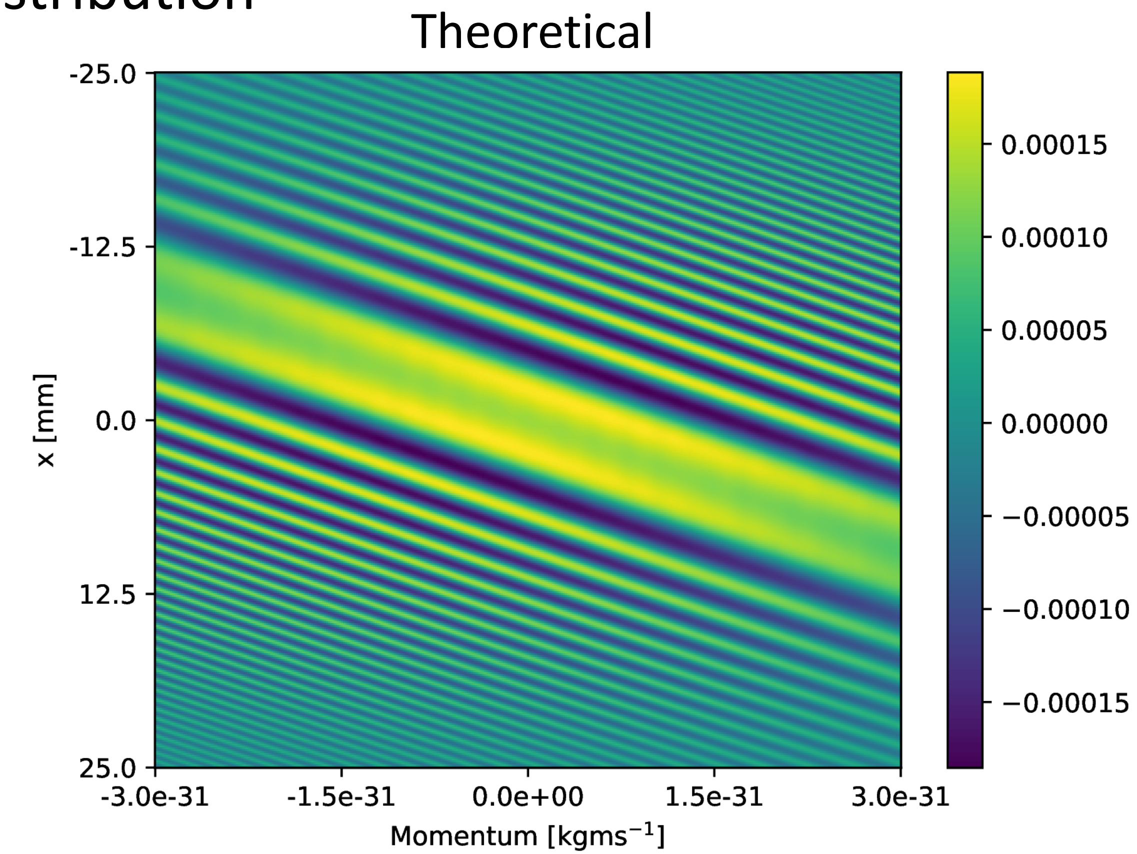
Results

Defocused Gaussian beam

As the amount of defocusing increases, the feature rotates within the distribution

Real part of KD distribution





Bailey, T., Weil, M., Lundeen, J., To be submitted.

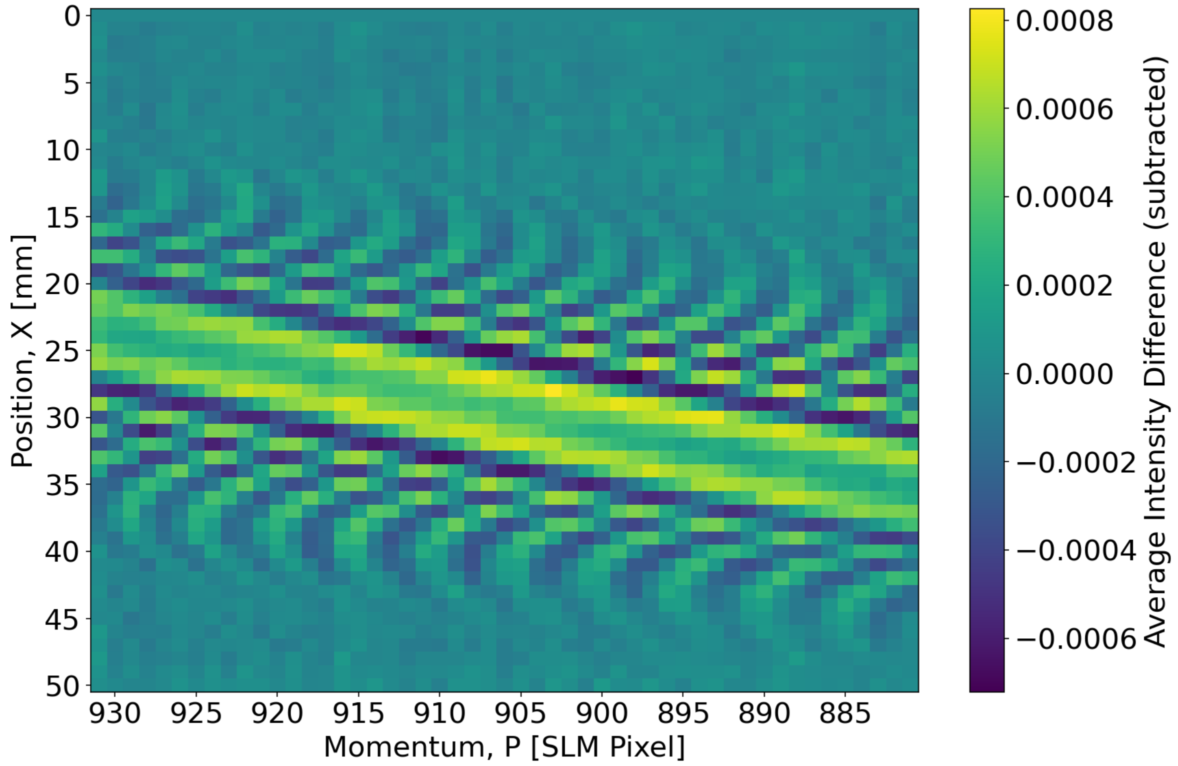
Results

Defocused Gaussian beam

Plotted at lower resolution to demonstrate aliasing effect between the expected fringes and low momentum resolution of the SLM

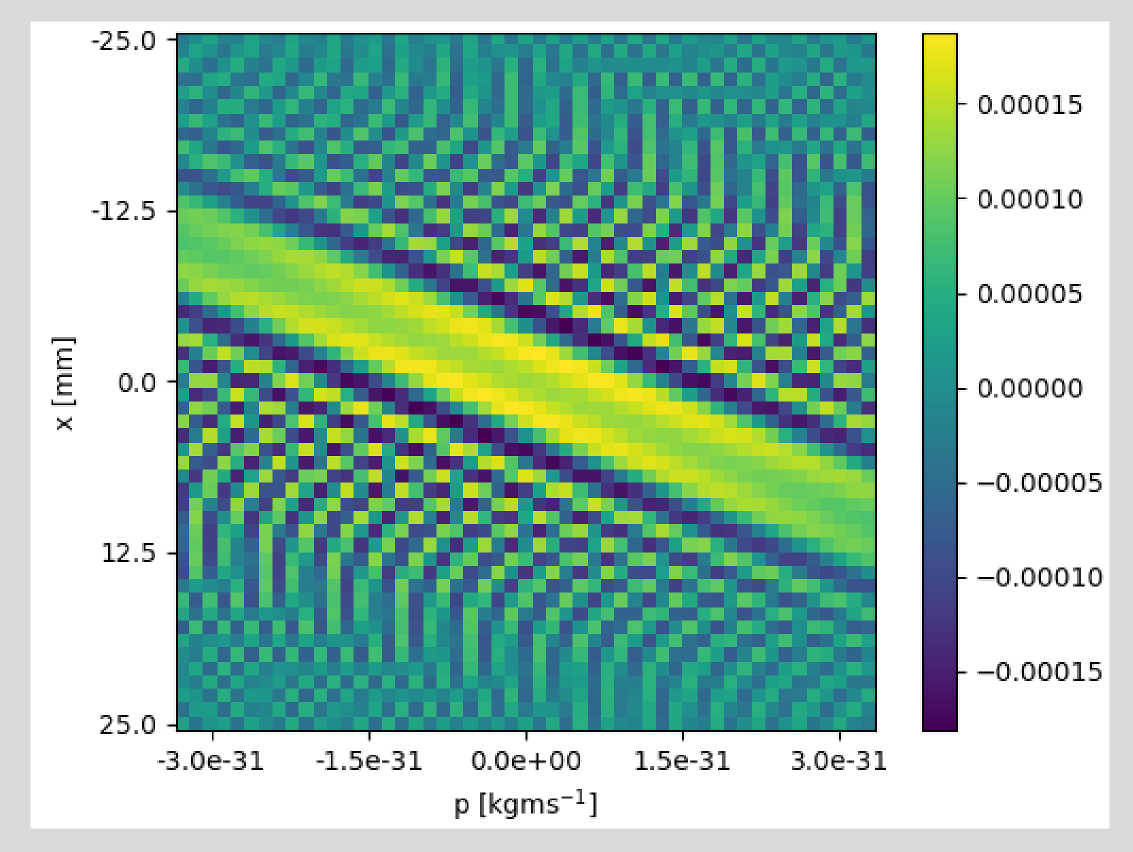
Real part of KD distribution

Experimental



Bailey, T., Weil, M., Lundeen, J., To be submitted.

Theoretical



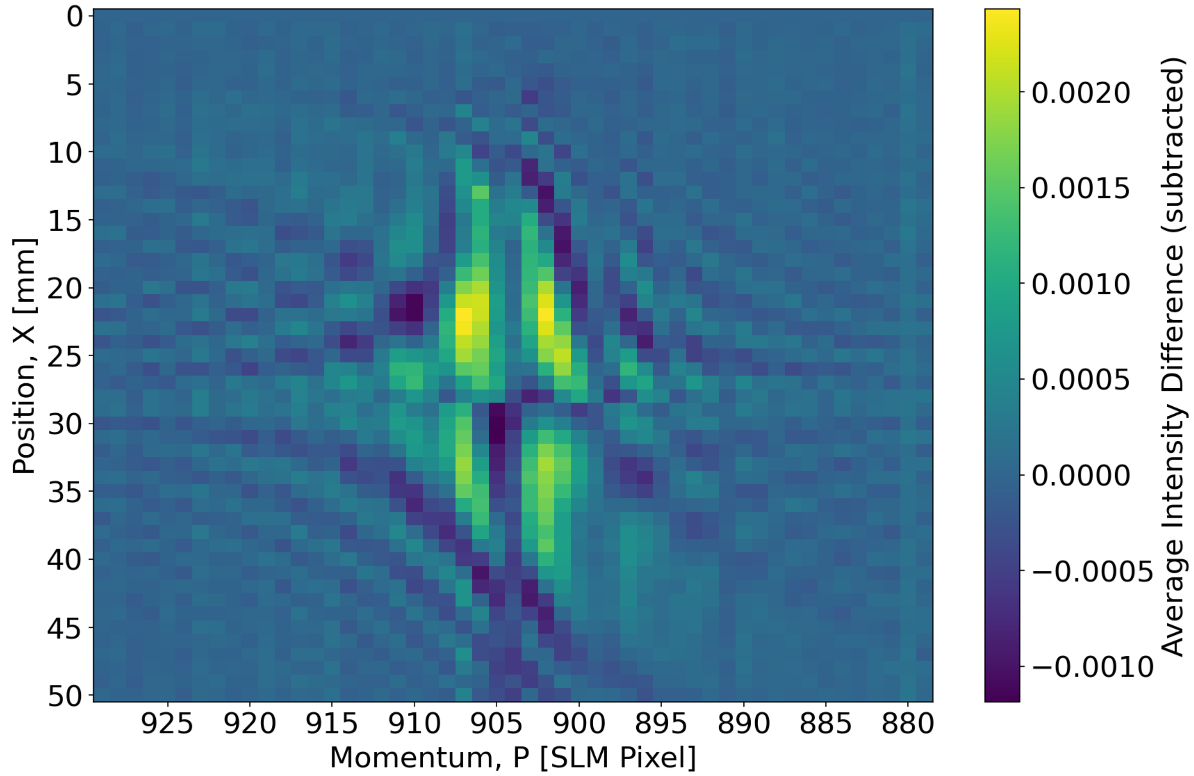
Bailey, T., Weil, M., Lundeen, J., To be submitted.

Results

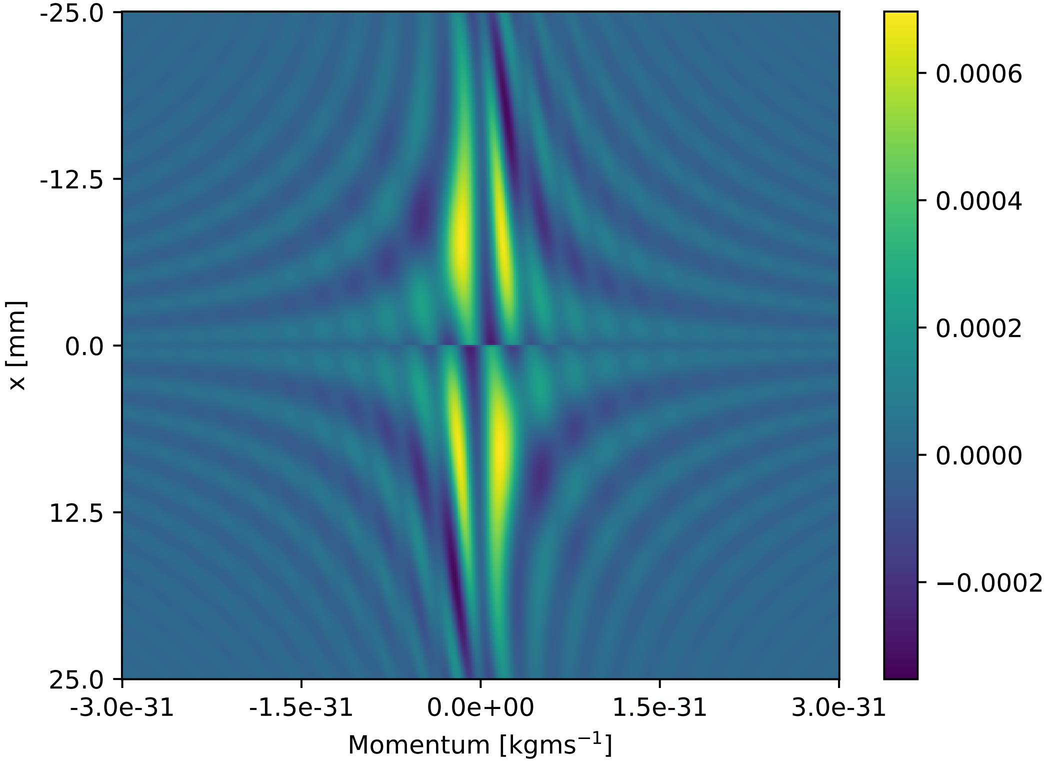
Truncated Gaussian beam + phase step

Real part of KD distribution

Experimental



Theoretical



Implementation: glass slide across half the beam imparts a sudden phase jump

Bailey, T., Weil, M., Lundeen, J., To be submitted.

Note: Imaginary part of joint value

To find the imaginary part of the joint value, the strength of the second interaction should depend on the first pointer's momentum (rather than position)

$$\hat{U} = \exp\left(-i(g_2\hat{p}_1)\hat{A}\hat{p}_2/\hbar\right) \times \exp\left(-ig_1\hat{B}\hat{p}_1/\hbar\right)$$

Experimentally:

Replace cylindrical FT lens with spherical FT lens. Why?

Fourier plane must now correspond to momentum for both x and the first pointer y.

$$\rightarrow \langle\hat{q}_2\rangle = \frac{\hbar g_1 g_2}{2\sigma_1^2}\mathrm{Im}\left[\langle\psi|\hat{A}\hat{B}|\psi\rangle\right]$$

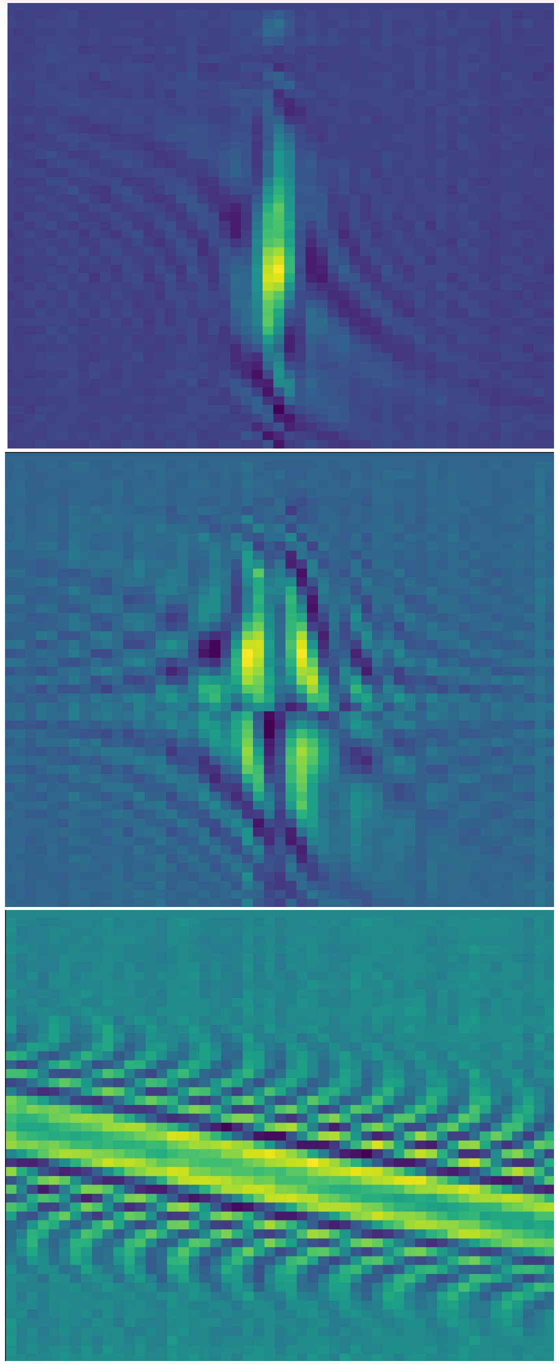
Where σ_1 is the standard deviation of the first pointer distribution

Summary

1. Conditional weak measurements of incompatible observables allows for the joint value to be read off from a single pointer

2. We demonstrate the use of conditional weak measurements to retrieve the KD distribution of various states

3. Future directions:
   a) Measure the imaginary part of the KD distribution
   b) Experimentally determine the density matrix of various states



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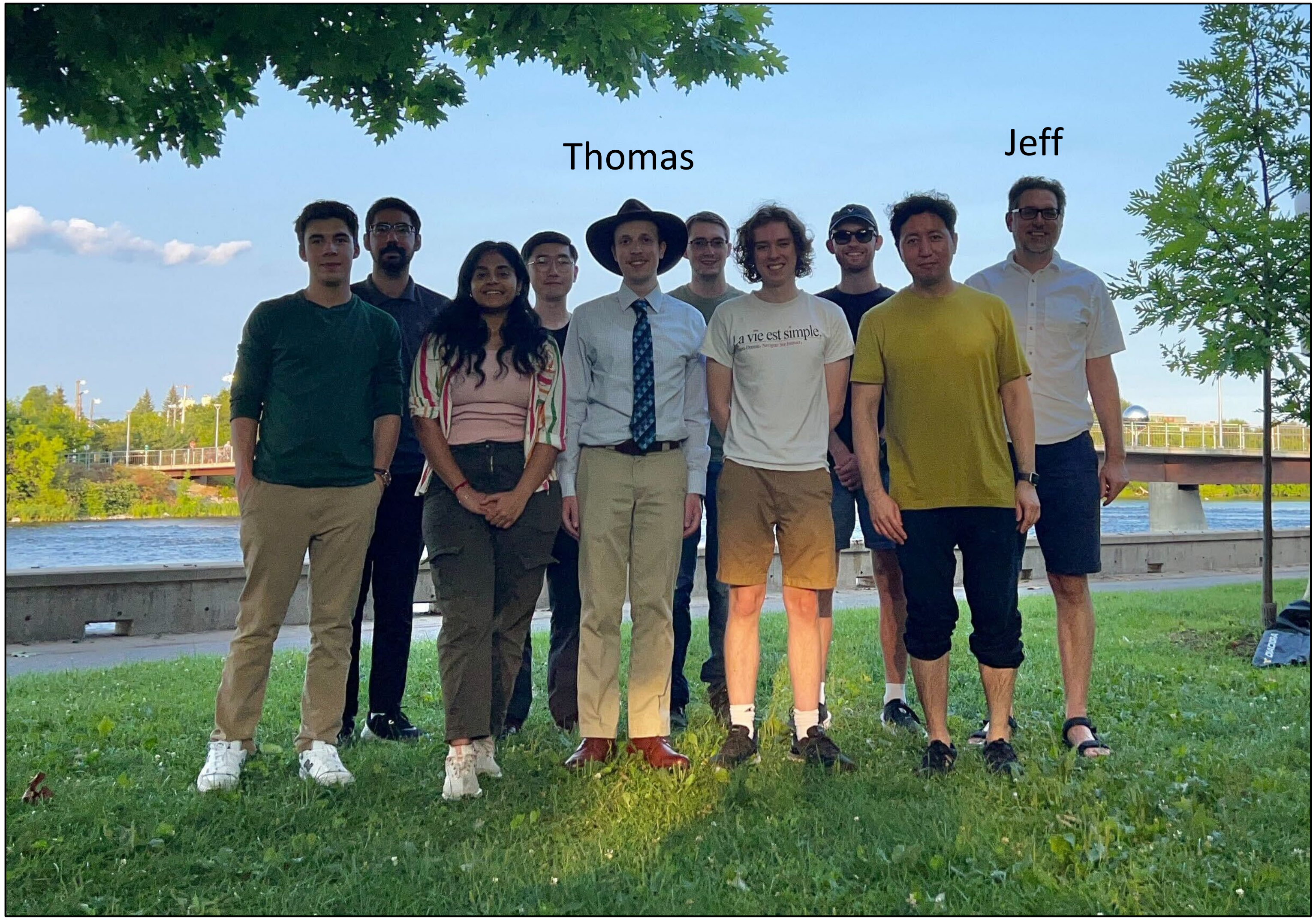

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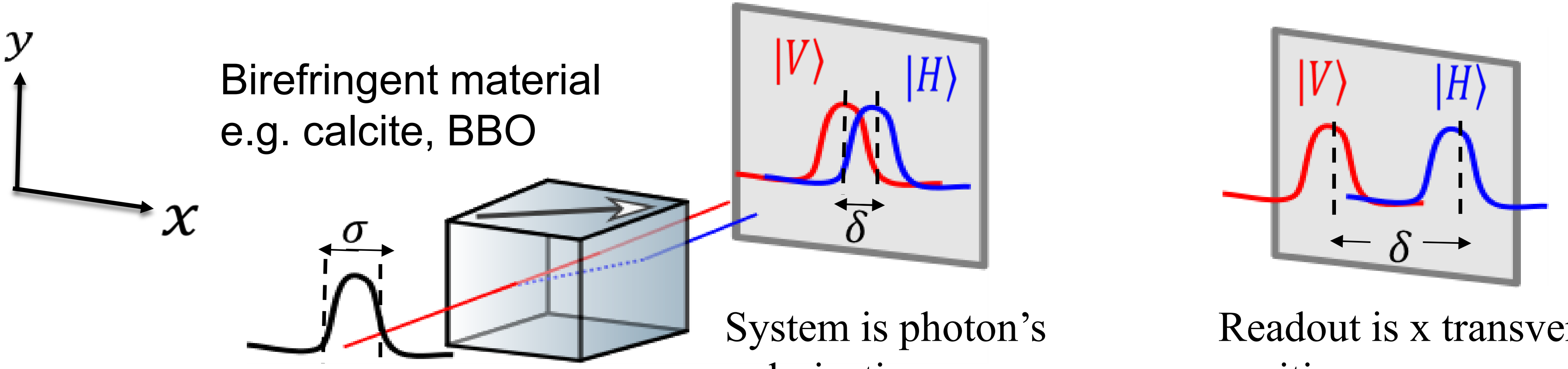
Example

Weak limit: $\delta \ll \sigma$

Strong limit: $\delta \gg \sigma$



System is photon's polarization.

Readout is x transverse position

Note: Imaginary part of joint value

Another option: the interaction with the first pointer can shift the pointer's momentum instead of position

$$\hat{U} = \exp\left(-i(g_2\hat{q}_1)\hat{A}\hat{p}_2/\hbar\right) \times \exp\left(-ig_1\hat{B}\hat{q}_1/\hbar\right)$$

$$\rightarrow \langle\hat{q}_2\rangle = \frac{2g_1g_2\sigma_1^2}{\hbar}\mathrm{Im}\left[\langle\psi|\hat{A}\hat{B}|\psi\rangle\right]$$