KD Quasiprobability Meets Bures Geometry:

Coherence - Predictability Duality and Triality

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Non-Classical Kirkwood-Dirac Coherence

Kirkwood-Dirac Quasiprobability

$$\mathrm{Pr}_{\mathrm{KD}}(a, b|\varrho) := \mathrm{Tr}(\Pi_b \Pi_a \varrho)$$

Budiyono et al., “Coherence from Kirkwood–Dirac nonclassicality,” *J. Phys. A* **57**, 255301 (2024).

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Non-Commutativity $\longrightarrow$ Quantumness $\longrightarrow$ Coherence

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$$C_{\mathrm{KD}}^{\mathrm{NCl}}(\varrho; \{\Pi_a\}) := \sup_{\{|b\rangle\} \in \mathcal{B}_{\mathrm{o}}(\mathcal{H})} \sum_{a,b} |\mathrm{Pr}_{\mathrm{KD}}(a, b|\varrho)| - 1$$

Budiyono et al., “Coherence from Kirkwood–Dirac nonclassicality,” *J. Phys. A* **57**, 255301 (2024).

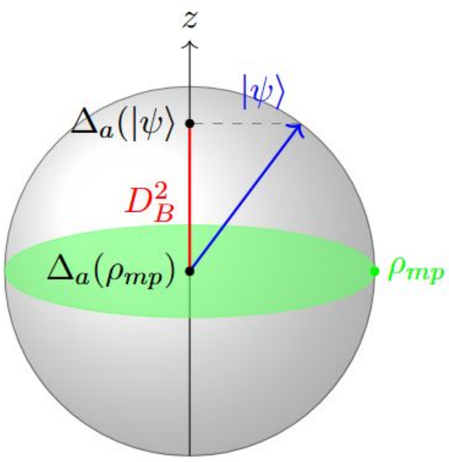
Dephased Bures Predictability

$$P(\rho, \{\Pi_a\}) := D_B^2(\Delta_a(\rho) \| \Delta_a(\rho_{mp}))$$

Maximally unpredictable state

Bures Distance

$$D_B^2(\rho\|\sigma) = 2\left(1 - \sqrt{F(\rho,\sigma)}\right),$$



Bloch Sphere

Fuchs & van de Graaf, "Cryptographic distinguishability measures for quantum states," *IEEE Trans. Inf. Theory* **45**(4), 1216–1227 (1999).

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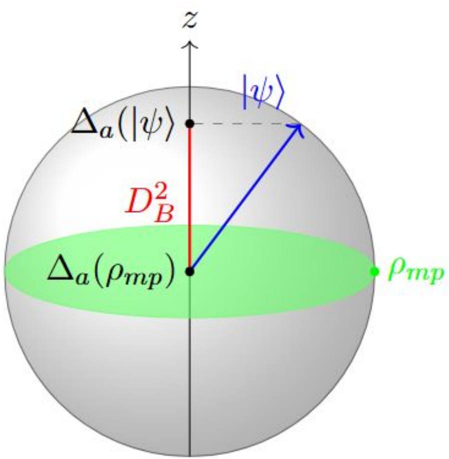

Bloch Sphere

For Pure States

$$\frac{C_{KD}^{NCl}(|\psi\rangle, \{\Pi_a\})}{\max_{|\psi\rangle} C_{KD}^{NCl}(|\psi\rangle, \{\Pi_a\})} = 1 - \frac{P(|\psi\rangle, \{\Pi_a\})}{\max_{|\psi\rangle} P(|\psi\rangle, \{\Pi_a\})}.$$

Fuchs & van de Graaf, “Cryptographic distinguishability measures for quantum states,” *IEEE Trans. Inf. Theory* **45**(4), 1216–1227 (1999).

Convex Roof Constructed Coherence

Fidelity by Uhlmann

$$F(|\psi\rangle, \rho) = \sqrt{\langle\psi|\rho|\psi\rangle} \Rightarrow F(\rho, \sigma) \equiv \mathrm{tr}\sqrt{\rho^{1/2}\sigma\rho^{1/2}}$$

Coherence

Pure States

$$C(\rho, \{\Pi_a\}) = \inf_{\{p_i, |\psi_i\rangle\}} \sum_i p_i C_{KD}^{NCl}(|\psi_i\rangle, \{\Pi_a\})$$

Uhlmann, “Roofs and convexity,” *Entropy* **12**, 1799–1832 (2010).

Wave-Particle Duality Relation (Quantum Duality)

$$\frac{C(\rho, \{\Pi_a\})}{\max_\rho C(\rho, \{\Pi_a\})} + \frac{P(\rho, \{\Pi_a\})}{\max_\rho P(\rho, \{\Pi_a\})} \leq 1.$$

Greenberger & Yasin, “Simultaneous wave & particle knowledge,” *Phys. Lett. A* **128**, 391–394 (1988).
Englert, “Fringe Visibility & Which-Way Information,” *Phys. Rev. Lett.* **77**, 2154–2157 (1996).

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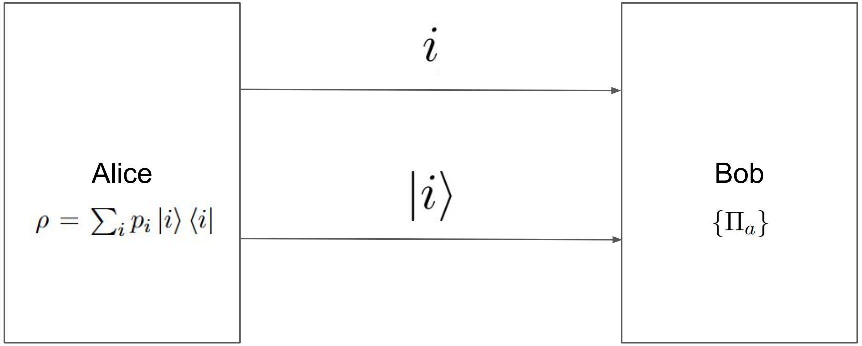
Coherence = Quantumness (Superposition)

Predictability = Particleness (Classical Determinism)

Greenberger & Yasin, "Simultaneous wave & particle knowledge," *Phys. Lett. A* **128**, 391–394 (1988).
Englert, "Fringe Visibility & Which-Way Information," *Phys. Rev. Lett.* **77**, 2154–2157 (1996).

Interpretation as Unpredictability



$$C(\rho, \{\Pi_a\}) = 1 - \sup_{\{p_i, |\psi_i\rangle\}} \sum_i p_i \frac{P(|\psi_i\rangle, \{\Pi_a\})}{\max_{|\psi\rangle} P(|\psi\rangle, \{\Pi_a\})}.$$

The maximum average predictability

Average Predictability of Bob's Measurement

Coherence = Irreducible Unpredictability

Triality Equation from Entropy Production

Upper Bound for Entropy Production

$$\Sigma_{1/2} \leq \frac{S_{1/2}(I/d)}{\sqrt{d}}$$

Triad Equality

$$\tilde{P}(\rho, \{\Pi_a\}) + \tilde{C}(\rho, \{\Pi_a\}) + \tilde{S}_{1/2}(\rho) = 1$$

Dephased Bures Predictability

(Tsallis-½) Relative Entropy of Coherence

Tsallis-½ Entropy

Xue et al., "Genuine quantum effects in nonequilibrium EP," *Phys. Rev. A* **110**, 042204 (2024).
Zhao & Yu, "Coherence measure via Tsallis relative α-entropy," *Sci. Rep.* **8**, 299 (2018).
Qian *et al*., "Turning off quantum duality," *Phys. Rev. Research* **2**, 012016 (2020).

Thank You

Resources

[1] Agung Budiyono, Joel F Sumbowo, Mohammad K Agusta, and Bagus E B Nurhandoko. Quantum coherence from kirkwood–dirac nonclassicality, some bounds, and operational interpretation. *Journal of Physics A: Mathematical and Theoretical*, 57(25):255301, jun 2024.

[2] Armin Uhlmann. Roofs and convexity. *Entropy*, 12(7):1799–1832, 2010.

[3] Daniel M. Greenberger and Allaine Yasin. Simultaneous wave and particle knowledge in a neutron interferometer. *Phys. Lett. A*, 128(8):391–394, 1988.

[4] Berthold-Georg Englert. Fringe Visibility and Which-Way Information: An Inequality. *Phys. Rev. Lett.*, 77:2154–2157, 1996.

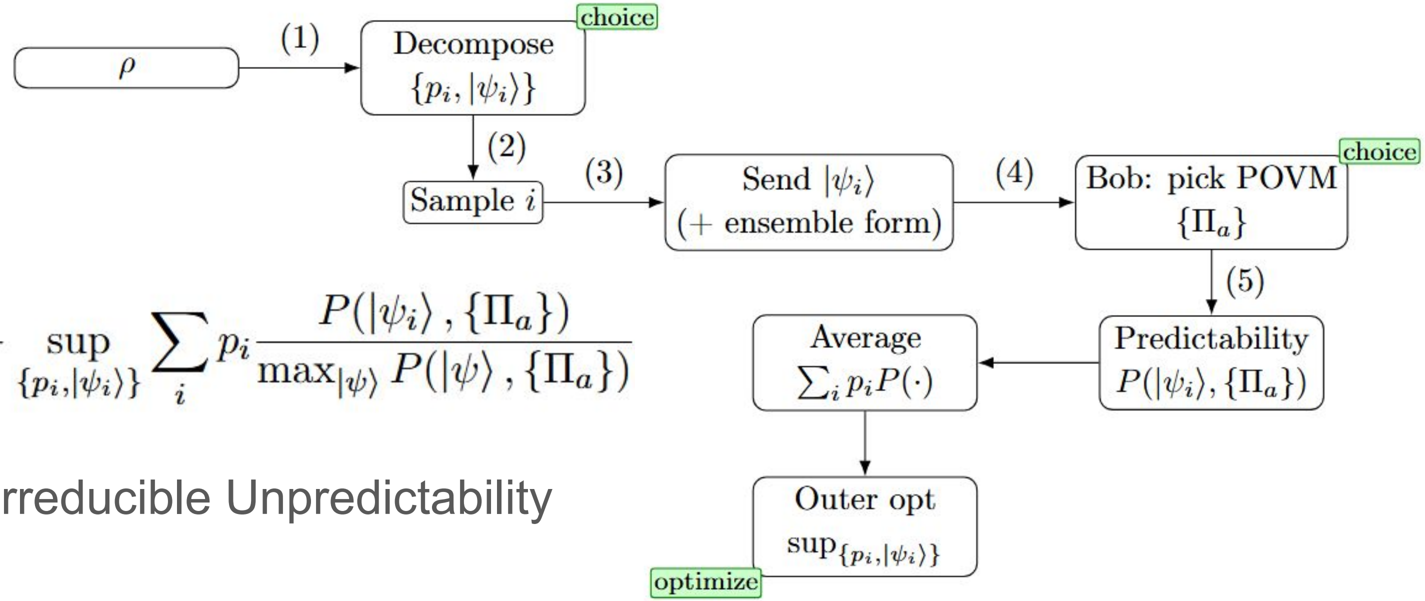
[5] Qing-Feng Xue, Xu-Cai Zhuang, De-Yang Duan, Ying-Jie Zhang, Wei-Bin Yan, Yun-Jie Xia, Rosario Lo Franco, and Zhong-Xiao Man. Evidence of genuine quantum effects in nonequilibrium entropy production via quantum photonics. *Phys. Rev. A*, 110(4):042204, 2024.

[6] X.-F. Qian, K. Konthasinghe, S. K. Manikandan, D. Spiecker, A. N. Vamivakas, and J. H. Eberly. Turning off quantum duality. *Phys. Rev. Res.*, 2:012016, Jan 2020.

[7] Claudio Carmeli, Teiko Heinosaari, and Alessandro Toigo. Quantum guessing games with posterior information. *Rept. Prog. Phys.*, 85:074001, 2022.

Appendices

Interpretation as Unpredictability



$$C(\rho, \{\Pi_a\}) = 1 - \sup_{\{p_i, |\psi_i\rangle\}} \sum_i p_i \frac{P(|\psi_i\rangle, \{\Pi_a\})}{\max_{|\psi\rangle} P(|\psi\rangle, \{\Pi_a\})}$$

Coherence = Irreducible Unpredictability

Predictability and Particle Definiteness as Resources

Coherence

Practical definition:
For a quantum state ρ, the coherence C(ρ, X) is any functional of the quantum state ρ which quantifies how coherent (how strong is the superposition of) the quantum state ρ is with respect to an incoherent basis X.

$$\rho_{\text{incoh}} = \begin{pmatrix} 0.50 & 0 & 0 & 0 \\ 0 & 0.30 & 0 & 0 \\ 0 & 0 & 0.15 & 0 \\ 0 & 0 & 0 & 0.05 \end{pmatrix} \qquad \rho_{\text{max-coh}} = \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

C = 0 C = 1

Predictability

Practical definition:
For a quantum state ρ written in the X basis, the predictability P(ρ, X) is any functional of the diagonal probabilities with respect to X which quantifies how certain the measurement is.

$$\rho_{\text{uniform}} = \frac{1}{4}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

P = 0

$$\rho_{\text{single}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

P = 1

Non Classical Kirkwood Dirac Coherence

$$C_{\mathrm{KD}}^{\mathrm{NCl}}(\varrho;\{\Pi_a\}) := \sup_{\{|b\rangle\}\in\mathcal{B}_{\mathrm{o}}(\mathcal{H})} \mathrm{NCl}(\{\mathrm{Pr}_{\mathrm{KD}}(a,b|\varrho)\})$$
$$= \sup_{\{|b\rangle\}\in\mathcal{B}_{\mathrm{o}}(\mathcal{H})} \sum_{a,b} \left| \langle b|\Pi_a\varrho|b\rangle \right| - 1,$$

$$\mathrm{NCl}(\{\mathrm{Pr}_{\mathrm{KD}}(a,b|\varrho)\}) := \sum_{a,b} |\mathrm{Pr}_{\mathrm{KD}}(a,b|\varrho)| - 1$$

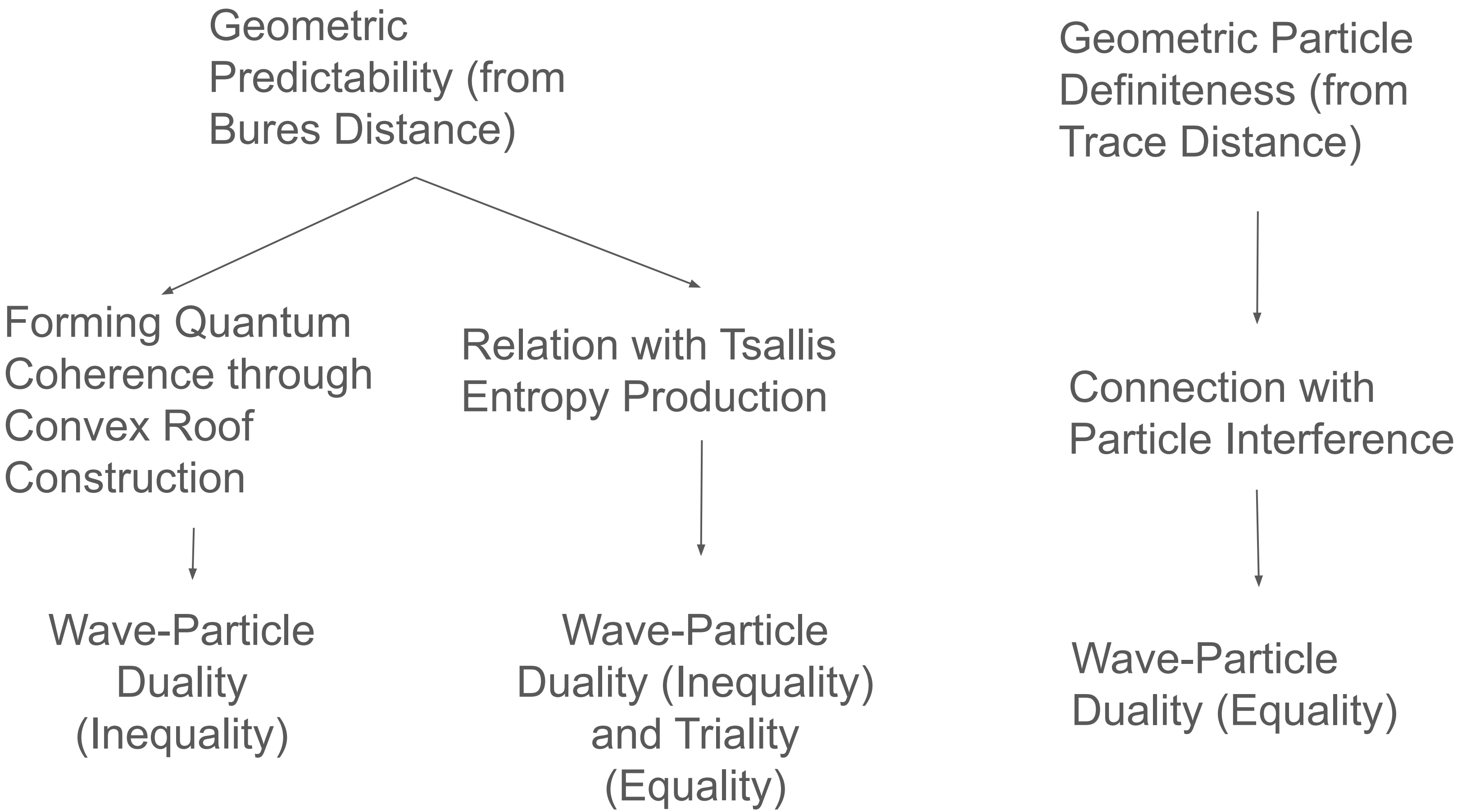
For Pure States $$C_{KD}^{NCl}(|\psi\rangle,\{\Pi_a\}) = -1 + \sum_a \sqrt{|\langle a|\psi\rangle|^2}$$

[1] Agung Budiyono, Joel F Sumbowo, Mohammad K Agusta, and Bagus E B Nurhandoko. Quantum coherence from kirkwood–dirac nonclassicality, some bounds, and operational interpretation. *Journal of Physics A: Mathematical and Theoretical*, 57(25):255301, jun 2024.

Bures Predictability

The two are related

$$\frac{C_{KD}^{NCl}(|\psi\rangle, \{\Pi_a\})}{\max_{|\psi\rangle} C_{KD}^{NCl}(|\psi\rangle, \{\Pi_a\})} = 1 - \frac{P(|\psi\rangle, \{\Pi_a\})}{\max_{|\psi\rangle} P(|\psi\rangle, \{\Pi_a\})}.$$



Geometric Predictability (from Bures Distance)
Forming Quantum Coherence through Convex Roof Construction
Relation with Tsallis Entropy Production
Wave-Particle Duality (Inequality)
Wave-Particle Duality (Inequality) and Triality (Equality)
Geometric Particle Definiteness (from Trace Distance)
Connection with Particle Interference
Wave-Particle Duality (Equality)

Geometric Coherence

Coherences that are defined using Fidelity.

$$C_g(\rho) = 1 - \max_{\sigma \in \mathcal{I}} F(\rho, \sigma),$$

$$C_g(\rho) = 1 - \max_{\sigma \in \mathcal{I}} \sqrt{F(\rho, \sigma)}.$$

Convex Roof Construction of a Coherence from the Geometric Predictability

Non-Classical Kirkwood-Dirac Coherence

$$C_{\mathrm{KD}}^{\mathrm{NCl}}(\varrho; \{\Pi_a\}) := \sup_{\{|b\rangle\} \in \mathcal{B}_o(\mathcal{H})} \sum_{a,b} \left| \langle b|\Pi_a \varrho|b\rangle \right| - 1,$$

Expression for Pure States

Bures Predictability

$$\frac{C_{KD}^{NCl}(|\psi\rangle, \{\Pi_a\})}{\max_{|\psi\rangle} C_{KD}^{NCl}(|\psi\rangle, \{\Pi_a\})} = 1 - \frac{D_B^2(\Delta_a(|\psi\rangle) \| \Delta_a(|\psi_{mc}\rangle))}{\max_{|\psi\rangle} D_B^2(\Delta_a(|\psi\rangle) \| \Delta_a(|\psi_{mc}\rangle))}.$$

Convex Roof Construction and Wave-Particle Duality Relation

Convex Roof Construction

$$C(\rho, \{\Pi_a\}) = \inf_{\{p_i, |\psi_i\rangle\}} \sum_i p_i C_{KD}^{NCl}(|\psi_i\rangle, \{\Pi_a\}).$$

Duality

$$\frac{C(\rho, \{\Pi_a\})}{\max_\rho C(\rho, \{\Pi_a\})} + \frac{P(\rho, \{\Pi_a\})}{\max_\rho P(\rho, \{\Pi_a\})} \leq 1.$$

Quantum Guessing Game with Classical Information

$$C(\rho, \{\Pi_a\}) = 1 - \sup_{\{p_i, |\psi_i\rangle\}} \sum_i p_i \frac{P(|\psi_i\rangle, \{\Pi_a\})}{\max_{|\psi\rangle} P(|\psi\rangle, \{\Pi_a\})}$$

The maximum average predictability

Average Predictability of Bob’s Measurement

Coherence quantifies the classically-irreducible unpredictability

Entropy Production

$$\Sigma = \Sigma_{classical} + \Sigma_{quantum}.$$

$$H(\rho\|\rho_{eq}) = H(\Delta_a(\rho)\|\rho_{eq}) + C_{rel}(\rho, \{\Pi_a\}).$$

Relative Entropy

Bures Predictability

Relative Entropy of Coherence

Q.-F. Xue et al., *Evidence of genuine quantum effects in nonequilibrium entropy production via quantum photonics*, Phys. Rev. A **110** (2024) 4, 042204, doi:10.1103/PhysRevA.110.042204, 110(4), 042204.

½-Tsallis Entropy Production

Upper Thermodynamic Bound
(Maximum Entropy Production)

$$\Sigma_{1/2} \leq \frac{S_{1/2}(I/d)}{\sqrt{d}}.$$

Duality

$$\tilde{P}(\rho, \{\Pi_a\}) + \tilde{C}(\rho, \{\Pi_a\}) \leq 1,$$

Triality (By setting the thermal eq state as the identity)

$$\tilde{P}(\rho, \{\Pi_a\}) + \tilde{C}(\rho, \{\Pi_a\}) + \tilde{S}_{1/2}(\rho) = 1,$$

Particle-Definiteness - Quantum Shell Game

Dichotomic Yes/No Measurement

$$\varrho_a := \Pi_a \varrho \Pi_a + (\mathbb{I} - \Pi_a)\varrho(\mathbb{I} - \Pi_a).$$

Trace Distance

Particle Definiteness

$$\mathrm{PD}(\varrho; \{\Pi_a\}) = 1 - \frac{\sum_a \|\varrho - \varrho_a\|_1}{\max_{\varrho \in \mathbb{D}(\mathcal{H})} \sum_a \|\varrho - \varrho_a\|_1},$$

Complementarity Relations

$$\mathrm{PD}(\varrho; \{\Pi_a\}) + \mathrm{PI}(\varrho; \{|a\rangle\}) = 1.$$

where

$$\mathrm{PI}(\varrho; \{|a\rangle\}) \ := \ \frac{C_{\mathrm{KD}}^{\mathrm{NRe}}(\varrho; \{|a\rangle\})}{\max_{\varrho \in \mathbb{D}(\mathcal{H})} C_{\mathrm{KD}}^{\mathrm{NRe}}(\varrho; \{|a\rangle\})}$$

Physical Significance of Wave-Particle Relation

- Express Bohr’s complementarity principle of wave and particle quantitatively through geometric measures.
- Turn off the duality by adding a third term (entropy) and turning the complementary relation into an equality which corresponds to recent results (adding entanglement).
- Relate two or more quantum resources and express their trade-off equations.

½-Tsallis Entropy Production

½-Tsallis Entropy

$$S_{1/2}(\rho) = 2\left(1 - \operatorname{Tr}\sqrt{\rho}\right),$$

Upper Thermodynamic Bound
(Maximum Entropy Production)

$$\Sigma_{1/2} \leq \frac{S_{1/2}(I/d)}{\sqrt{d}}.$$

$$H_{1/2}(\Delta(\rho)\|I/d) + C_{rel,1/2}(\rho, \{\Pi_a\}) \leq \frac{S_{1/2}(I/d)}{\sqrt{d}}.$$

Wave-Particle Triality Relation

Relation between ½-Tsallis Relative Entropy to Normalised Bures Predictability

$$\sqrt{d}\frac{H_{1/2}(\Delta(\rho)\|/d)}{S_{1/2}(I/d)} = \tilde{P}(\rho, \{\Pi_a\}),$$

Duality

$$\tilde{P}(\rho, \{\Pi_a\}) + \tilde{C}(\rho, \{\Pi_a\}) \leq 1,$$

Triality (By setting the thermal eq state as the identity)

$$\tilde{P}(\rho, \{\Pi_a\}) + \tilde{C}(\rho, \{\Pi_a\}) + \tilde{S}_{1/2}(\rho) = 1,$$

Particle Interference - Coherence

Non-real Kirkwood Dirac Coherence

$$C_{\mathrm{KD}}[\varrho;\{\Pi_a\}] \;:=\; \max_{\{|b\rangle\}} \sum_a \sum_b \frac{1}{2} \left| \langle b|[\Pi_a,\varrho]|b\rangle \right|,$$

$$\begin{aligned}\mathrm{PI}(\varrho;\{|a\rangle\}) \;&:=\; \frac{C_{\mathrm{KD}}^{\mathrm{NRe}}(\varrho;\{|a\rangle\})}{\max_{\varrho\in\mathbb{D}(\mathcal{H})} C_{\mathrm{KD}}^{\mathrm{NRe}}(\varrho;\{|a\rangle\})} \\ &= \frac{C_{\mathrm{KD}}^{\mathrm{NRe}}(\varrho;\{|a\rangle\})}{\sqrt{d-1}}.\end{aligned}$$