Efficient classical simulation of quantum computation beyond Wigner nonnegativity

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November 7, 2025

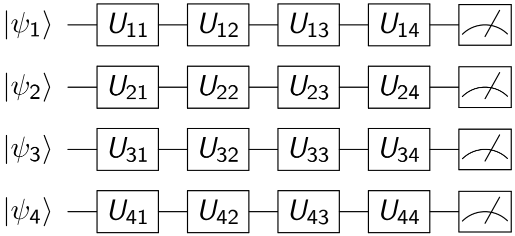
The big question

What is the essential quantum resource that provides the computational speedup over classical computation?

We approach this problem through classical simulation:

- If a family of quantum circuits can be efficiently simulated classically, then it offers no quantum computational advantage

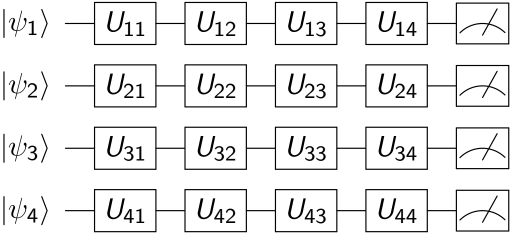
Motivating example



Consider a quantum circuit consisting of

1. preparation of a product state $|\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots$,
2. a sequence of local (single-qudit) unitary gates,
3. a measurement of each qudit in the computational basis.

Motivating example



This can be simulated efficiently on a classical computer. Therefore,

entanglement is necessary for quantum computational advantage.

Stabilizer subtheory

1. Pauli measurements, $\mathcal{P} = \langle X, Y, Z \rangle$
   - For any $a = (a_z, a_x) \in \mathbb{Z}_d^n \times \mathbb{Z}_d^n$,

$$T_a := e^{i\phi(a)} \bigotimes_{k=1}^{n} Z^{a_z[k]} X^{a_x[k]}$$

   - Eigenvalues are $\omega^s,\ s \in \mathbb{Z}_d$, where $\omega = \exp(2\pi i/d)$
   - Π_a^s is a projector corresponding to measurement T_a giving outcome $s \in \mathbb{Z}_d$

2. Clifford gates, $\mathcal{C}\ell = \mathcal{N}(\mathcal{P})/U(1)$
   - For any $g \in \mathcal{C}\ell$,

$$g T_a g^\dagger = \omega^{\Phi_g(a)} T_{g \cdot a}$$

3. Stabilizer states, $\mathcal{S}$
   - Eigenstates of Pauli operators

Gottesman-Knill theorem[1]

Theorem

Any quantum circuit consisting of

1. *preparation of stabilizer states,*
2. *Clifford unitary gates,*
3. *Pauli measurements*

can be efficiently simulated classically.

- Quantum computational universality is restored in the circuit model by supplementing the stabilizer subtheory with non-Clifford gates (e.g. T gates)

[1] Gottesman. arXiv:9807006 (1998)

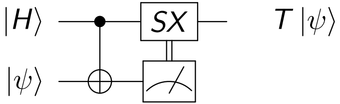
Quantum computation with magic states (QCM)[2]

$|magic\rangle \Rightarrow$
$|magic\rangle \Rightarrow$
$|magic\rangle \Rightarrow$

$\boxed{\text{Stabilizer operations: Clifford gates+ Pauli Measurements}} \Rightarrow$ Output

- Stabilizer operations alone $\Rightarrow$ efficiently simulable
- Stabilizer operations $+$ magic states $\Rightarrow$ universal

$$|H\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + e^{i\pi/4}|1\rangle\right)$$



[2]Bravyi and Kitaev. Phys Rev A (2005)

Discrete Wigner function[3,4]

- Phase space is $\mathbb{Z}_d^{2n}$
- Displacement operators are Pauli operators (or Heisenberg-Weyl operators)
- Parity operator is a sum of all Pauli operators

$$A_0 = \sum_{v \in \mathbb{Z}_d^{2n}} T_v$$

- Phase space point operators are

$$A_u = T_u A_0 T_u^\dagger, \quad \forall u \in \mathbb{Z}_d^{2n}$$

- The Wigner function $W_\rho : \mathbb{Z}_d^{2n} \to \mathbb{R}$ of a state ρ is

$$W_\rho(u) = \frac{1}{d^n} \mathrm{Tr}(\rho A_u)$$

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[3] Gross. Ph.D thesis Imperial College London (2005)

[4] Veitch, Ferrie, Gross, Emerson. New J Phys (2012)

Odd-dimensional Wigner function

Phase space points are identified with noncontextual value assignments (NCVAs) on Pauli observables:

functions $\gamma : \mathbb{Z}_d^{2n} \to \mathbb{Z}_d$ satisfying:

$$\omega^{-\gamma(a)-\gamma(b)} T_a T_b = \omega^{-\gamma(a+b)} T_{a+b} \quad \forall \text{ commuting } a, b.$$

Let V be the set of noncontextual assignments on $\mathbb{Z}_d^{2n}$

For each $\gamma \in V$ define a phase space point operator

$$A^\gamma = \frac{1}{d^n} \sum_{b \in \mathbb{Z}_d^{2n}} \omega^{-\gamma(b)} T_b$$

Equivalence of Wigner function negativity and contextuality[5]

- For $n \geq 2$, these definitions are equivalent

$$A_u = T_u A_0 T_u^\dagger = \sum_{v \in \mathbb{Z}_d^{2n}} T_u T_v T_u^\dagger = \sum_{v \in \mathbb{Z}_d^{2n}} \omega^{[u,v]} T_v$$

- The function $\gamma_u(\cdot) = -[u, \cdot]$ is a NCVA on $\mathbb{Z}_d^{2n}$

- Consistency constraints force all NCVAs on $\mathbb{Z}_d^{2n}$ to have this form

[5] Delfosse, Okay, Bermejo-Vega, Browne, Raussendorf. New J Phys **19** 123024 (2017)

Odd-dimensional Wigner function

1. Phase space point operators are

$$A^{\gamma} = \frac{1}{d^n} \sum_{b \in \mathbb{Z}_d^{2n}} \omega^{-\gamma(b)} T_b, \quad \forall \gamma \in V.$$

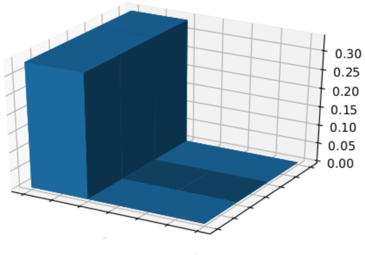
2. $\{A^{\gamma} \mid \gamma \in V\}$ is a basis for the space of Hermitian operators on Hilbert space

3. The Wigner function is defined by

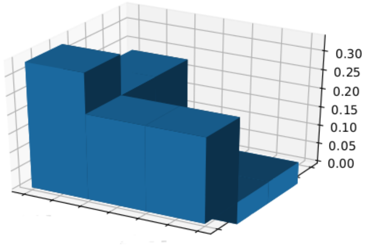
$$\rho = \sum_{\gamma \in V} W_{\rho}(\gamma) A^{\gamma}$$

Example Wigner functions[6,7]

$$\rho = \sum_{\gamma \in V} W_\rho(\gamma) A^\gamma$$



(a) Stabilizer state



(b) Magic state

Discrete Hudson's theorem: stabilizer states have nonnegative Wigner function

[6] Gross. J Math Phys (2006)
[7] Anwar, Campbell, Browne. New J Phys (2012)

Quasiprobability representations for quantum computation

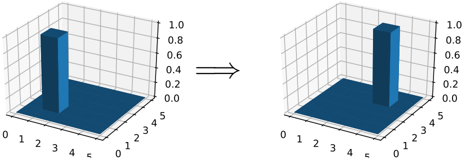
States

$$\rho = \sum_{\gamma \in \mathcal{V}} W_\rho(\gamma) A^\gamma$$

with $\sum_\gamma W_\rho(\gamma) = 1$

Clifford gates:

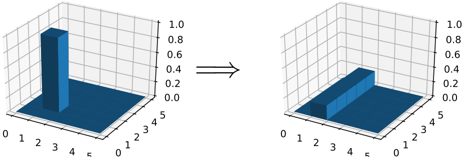
$$g A^\gamma g^\dagger = A^{g \cdot \gamma}$$



Pauli measurements:

$$\Pi_a^s A^\gamma \Pi_a^s = \sum_{\gamma' \in \mathcal{V}} q_{\gamma,a}(\gamma', s) A^{\gamma'}$$

with $q_{\gamma,a} \geq 0$, $\sum q_{\gamma,a}(\gamma', s) = 1$



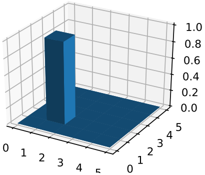
Classical Simulation Algorithm ($W \geq 0$)

When $W_\rho \geq 0$, it is a probability distribution.

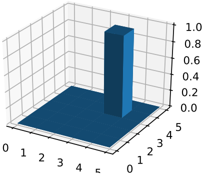
$\rho \Rightarrow$ **Stabilizer operations: Clifford gates+ Pauli Measurements** $\Rightarrow$ Output

1. Sample phase space points A^γ according to $W_\rho(\gamma)$,
2. Propagate phase space points through the circuit.
   Return definite values for measurements according to γ.

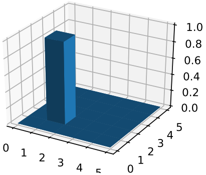
Update under Clifford gates:



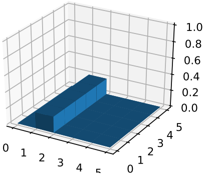
$\Rightarrow$



Update under Pauli measurements:



$\Rightarrow$



Bound magic states

- There exist nonstabilizer states with $W_\rho \geq 0$
- These are called bound magic states

Wigner negativity is necessary for quantum computational advantage
and this is a strictly stronger criterion than being nonstabilizer

No-go results for existence of Wigner functions

1. No Wigner function for even-dimensional qudits is Clifford covariant[8,9]
2. No Wigner function for even-dimensional qudits is positivity preserving under Pauli measurements[8,9]
3. Memory lower bound $\frac{1}{2}n(n-1)$ bits required to simulate contextuality[10]
   $\Rightarrow$ No qubit Wigner function can have a phase space where the phase point operators form an operator basis

[8] Raussendorf, Okay, MZ, Feldmann. Quantum 7, 979 (2023)
[9] Schmid, Du, Selby, Pusey. Phys Rev Lett (2022)
[10] Karanjai, Wallman, Bartlett. arXiv:1802.07744

The trouble with qubits[11]

- For qudits, phase space points are identified with noncontextual value assignements for Pauli observables
- For qubits, we have state-independent proofs of contextuality like Mermin square
- $\Rightarrow$ Noncontextual assignments for multiple qubits do not exist

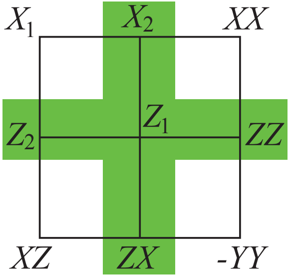
$$
\begin{array}{ccccc}
X_1 & — & X_2 & — & X_1X_2 \\
| & & | & & | \\
Z_2 & — & Z_1 & — & Z_1Z_2 \\
| & & | & & | \\
X_1Z_2 & — & Z_1X_2 & — & Y_1Y_2
\end{array}
$$

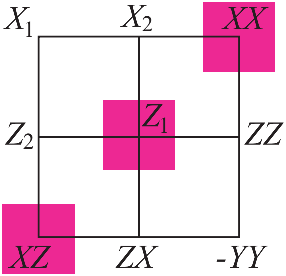
[11] Mermin. Rev Mod Phys (1993)

CNC construction for qubits[12]

We restrict to subsets $\Omega \subset \mathbb{Z}_d^{2n}$ of Pauli operators satisfying:

- If $a, b \in \Omega$ with $T_a T_b = T_b T_a$, then $a + b \in \Omega$
- There exists a noncontextual value assignment for Ω





Phase space point operators:

$$A_\Omega^\gamma = \frac{1}{d^n} \sum_{b \in \Omega} \omega^{-\gamma(b)} T_b$$

Wigner function:

$$\rho = \sum_{(\Omega,\gamma)} W(\Omega, \gamma) A_\Omega^\gamma$$

[12] Raussendorf, Bermejo-Vega, Tyhurst, Okay, MZ. Phys Rev A (2020)

CNC construction for qubits

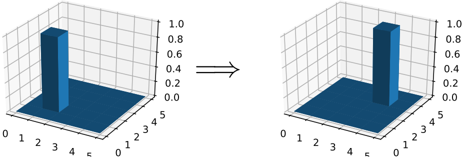
States

$$\rho = \sum_{(\Omega,\gamma)} W_\rho(\Omega,\gamma) A_\Omega^\gamma$$

with $\sum_{(\Omega,\gamma)} W_\rho(\Omega,\gamma) = 1$

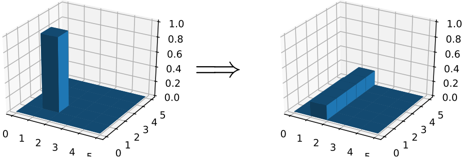
Clifford gates:

$$g A_\Omega^\gamma g^\dagger = A_{g\cdot\Omega}^{g\cdot\gamma}$$



Pauli measurements:

$$\Pi_a^s A_\Omega^\gamma \Pi_a^s = \begin{cases} \frac{1}{2}\left(A_\Omega^\gamma + A_\Omega^{\gamma+[a,\cdot]}\right) & \text{if } a \in \Omega \\ A_{\Omega\times a}^{\gamma\times a} & \text{if } a \notin \Omega \end{cases}$$



Examples of CNC operators

CNC operators

- Ω is a CNC set
- γ is a NCVA on Ω

$$A_\Omega^\gamma = \frac{1}{d^n} \sum_{b \in \Omega} \omega^{-\gamma(b)} T_b$$

Odd d phase space point operators

For a phase space point $u \in \mathbb{Z}_d^{2n}$,

- $\Omega = \mathbb{Z}_d^{2n}$
- $\gamma_u(\cdot) = -[u, \cdot]$

$$A^{\gamma_u} = \frac{1}{d^n} \sum_{b \in \mathbb{Z}_d^{2n}} \omega^{-\gamma_u(b)} T_b$$

Stabilizer states

For a stabilizer state $|\sigma\rangle$ with stabilizer group $\mathcal{S} = \{\omega^{r(a)} T_a | a \in I\}$,

- $\Omega = I$ is a lagrangian subspace of $\mathbb{Z}_d^{2n}$
- r is a NCVA on I

$$|\sigma\rangle \langle\sigma| = A_I^r = \frac{1}{d^n} \sum_{a \in I} \omega^{-r(a)} T_a$$

Multiqubit CNC operators can be classified

For any CNC operator A_Ω^γ,

$$A_\Omega^\gamma = g(A_{\tilde{\Omega}}^{\tilde{\gamma}} \otimes |\sigma\rangle\langle\sigma|)g^\dagger$$

where

1. g is a Clifford unitary, $|\sigma\rangle$ is a stabilizer state
2. $A_{\tilde{\Omega}}^{\tilde{\gamma}} = \frac{1}{2^n} \sum_{b \in \tilde{\Omega}} (-1)^{\tilde{\gamma}(b)} T_b \quad \text{with} \quad \{T_a, T_b\} = 2\delta_{a,b} \quad \forall a, b \in \tilde{\Omega}$

Equivalently,

$$\Omega = \bigcup_{k=1}^{\xi} \langle a_k, I \rangle$$

where I is an isotropic subspace of $\mathbb{Z}_2^{2n}$, $[a_k, g] = 0$ for all $g \in I$, and $[a_i, a_j] \neq 0$.

We only need maximal CNC sets for qubits

- Suppose Ω, $\bar{\Omega}$ are CNC sets with $\Omega \subsetneq \bar{\Omega}$
- For any NCVA γ on Ω, we can find two NCVAs, γ_0, γ_1, on $\bar{\Omega}$ such that
  - $\gamma_0(a) = \gamma_1(a) = \gamma(a) \quad \forall a \in \Omega$
  - $\gamma_1(a) \equiv \gamma_1(a) + 1 \mod 2 \quad \forall a \in \bar{\Omega} \setminus \Omega$
- Then $\frac{1}{2} A_{\bar{\Omega}}^{\gamma_0} + \frac{1}{2} A_{\bar{\Omega}}^{\gamma_1} = A_{\Omega}^{\gamma}$
- So for any state

$$\rho = \sum_{(\Omega,\gamma)} W_\rho(\Omega,\gamma) A_\Omega^\gamma$$

if $\exists W_\rho(\Omega,\gamma) > 0$ where Ω is nonmaximal, we can substitute $\frac{1}{2} A_{\bar{\Omega}}^{\gamma_0} + \frac{1}{2} A_{\bar{\Omega}}^{\gamma_1}$ for A_Ω^γ without increasing negativity.

CNC construction for qudits

- For odd-prime dimensional qudits, the only maximal CNC set is $\mathbb{Z}_d^{2n}$, i.e., the set of all Pauli observables

- For multiple qudits, CNC phase space point operators with $\Omega = \mathbb{Z}_d^{2n}$ are exactly the Wigner phase space point operators

- If the previous result holds for qudits too, then CNC construction is equivalent to the Wigner function

An explicit counter-example

- For a single qutrit
$$\mathbb{Z}_3^2 = \langle x \rangle \cup \langle z \rangle \cup \langle x+z \rangle \cup \langle x+2z \rangle$$
- There are 9 linear functions on $\mathbb{Z}_3^2$ (Wigner function phase space points)
- There are 81 NCVAs (CNC phase space points)

- For any single-qutrit CNC pair (Ω_1, γ_1) where γ_1 is nonlinear, any n-qutrit Clifford gate g, and any $n-1$-qutrit stabilizer state $|\sigma\rangle$,
$$g(A_{\Omega_1}^{\gamma_1} \otimes |\sigma\rangle\langle\sigma|)g^\dagger = A_\Omega^\gamma$$
is CNC where γ cannot be extended to a NCVA on $\mathbb{Z}_3^{2n}$.

Multiqudit CNC operators can be classified

Theorem

For any number of qudits n of any odd prime dimension d, a set $\Omega \subset \mathbb{Z}_d^{2n}$ is closed under inference if and only if

(i) Ω *is a subspace of* $\mathbb{Z}_d^{2n}$, *or*

(ii) Ω *has the form*

$$\Omega = \bigcup_{k=1}^{\xi} \langle a_k, I \rangle$$

where $I \subset E$ is an isotropic subspace, $[g, a_i] = 0$ for all $g \in I$, and $[a_i, a_j] \neq 0$.

The CNC construction describes quantum computations

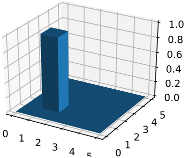
States

$$\rho = \sum_{(\Omega,\gamma)} W_\rho(\Omega,\gamma) A_\Omega^\gamma$$

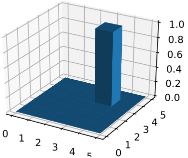
with $\sum_{(\Omega,\gamma)} W_\rho(\Omega,\gamma) = 1$

Clifford gates:

$$g A_\Omega^\gamma g^\dagger = A_{g\cdot\Omega}^{g\cdot\gamma}$$

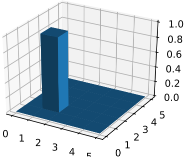


$\Longrightarrow$

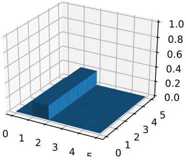


Pauli measurements:

$$\Pi_a^s A_\Omega^\gamma \Pi_a^s \propto A_{\langle a\rangle + \Omega\cap a^\perp}^{\gamma\times s}$$



$\Longrightarrow$



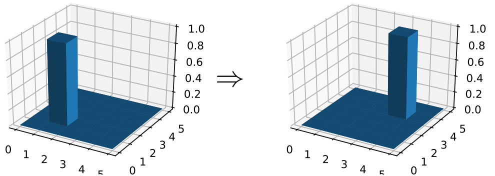
Classical Simulation Algorithm ($W \geq 0$)

When $W_\rho \geq 0$, it is a probability distribution.

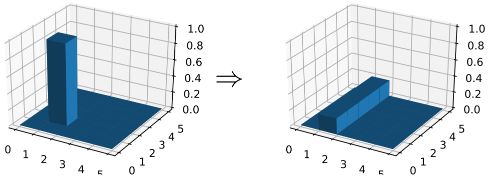
$$\rho \Rightarrow \boxed{\begin{array}{c}\text{Stabilizer operations:}\\ \text{Clifford gates+}\\ \text{Pauli Measurements}\end{array}} \Rightarrow \text{Output}$$

1. Sample phase space points A_Ω^γ according to $W_\rho(\Omega, \gamma)$,
2. Propagate phase space points through the circuit.
   Return definite values for measurements according to γ if $a \in \Omega$,
   return uniformly random outcomes otherwise.

Update under Clifford unitaries:



Update under Pauli measurements:



Summary

- Wigner function negativity is necessary for quantum computational advantage on odd-dimensional qudits
- Wigner functions don't exist for qubits
- CNC construction works for any dimension, including qubits
- For odd dimensions, CNC provides a new classical simulation algorithm that can simulate some quantum computations on Wigner negative states