PADERBORN UNIVERSITY

QuiDiQua^3 | 2025–11–06 | Paris

Quasiprobabilities from incomplete and overcomplete measurements

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PhoQS
INSTITUTE FOR PHOTONIC QUANTUM SYSTEMS

- interdisciplinary department for photonic quantum technologies
  - ✓ experimental physics
  - ✓ theoretical physics
  - ✓ mathematics
  - ✓ computer science
  - ✓ electrical engineering
- topics: quantum computation, quantum communication, quantum sensing, quantum simulation, photonic quantum systems, ...

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speaker

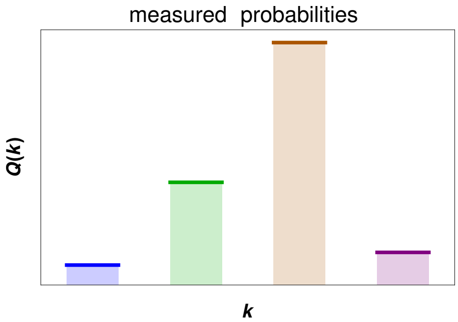
Christine Silberhorn

Motivation

- ✓ quantum tomography and state reconstruction & quasiprobability representation

$$\hat{\rho} = \begin{pmatrix} 0.83 & 0.02 - i\,0.35 \\ 0.02 + i\,0.35 & 0.17 \end{pmatrix}$$

↕ ↕



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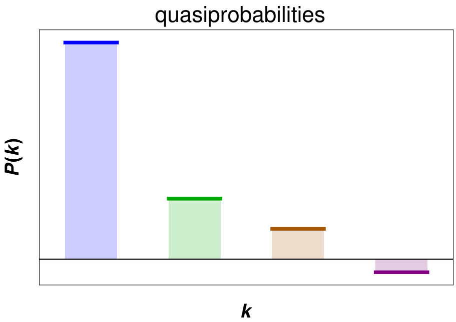

Motivation

? what if measured data $\vec{Q} = [Q(k)]_k$ are

informationally **incomplete**, failing **full** reconstruction

informationally **overcomplete**, failing **unique** reconstruction

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do not describe all parts of the quantum state $\hat{\rho}$

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↷ goal. construction of *measurement-based quasiprobabilities*

Formalism

- premise. consider informationally complete measurement operators $\{\hat{\Pi}_k\}_k$
  and outcomes $Q(k) = \mathrm{tr}\big(\hat{\Pi}_j^\dagger \hat{\rho}\big)$
- metric tensor $g_{j,k} = \mathrm{tr}\big(\hat{\Pi}_j^\dagger \hat{\Pi}_k\big)$ $\rightarrow$ yields relation:

$$\vec{Q} = \boldsymbol{g}\vec{P} \quad \rightarrow \quad \vec{P} = \boldsymbol{g}^{-1}\vec{Q}$$

Def. $\vec{P} \geq 0$ if state non-negative mixture
of measurement operators, $\boxed{\hat{\rho} = \sum_j P(j)\hat{\Pi}_j}$

↷ quantum superpositions via $\vec{P} \not\geq 0$

Formalism

- partial inversion $0 \leq \sigma \leq 1$:

$$\vec{P}_\sigma = \boldsymbol{g}^{-\sigma} \vec{Q}$$

(e.g., $\vec{P}_0 = \vec{Q}$ and $\vec{P}_1 = \vec{P}$)

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- $\boldsymbol{g}$ not invertible (overcomplete) $\rightarrow$ pseudo-inverse $\boldsymbol{g}^{-1}$

$$\vec{P}_\sigma = \boldsymbol{g}^{-\sigma} \vec{Q} \boxed{+\vec{N}}, \quad \text{with} \quad \boldsymbol{g}\vec{N} = 0$$

optimization over all possible $\vec{N}$

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optimization over all possible $\vec{N}$

- non-observed components, $\hat{\nu} \neq 0$ s.t. $\forall k : \mathrm{tr}\left(\hat{\Pi}_k^\dagger \hat{\nu}\right) = 0$ (incomplete)

$$\rightarrow \quad \hat{\rho} = \sum_k P(k)\hat{\Pi}_k \boxed{+\hat{\nu}}$$

nonclassicality agnostic w.r.t. $\hat{\nu}$

Elementary examples

✓ *Kirkwood–Dirac distributions*

weak measurements $\hat{\Pi}_{(a,b)} = \dfrac{|a\rangle\langle b|}{\langle b|a\rangle}$

for orthonormal bases $\mathcal{A}$ and $\mathcal{B}$

with $|a\rangle \in \mathcal{A}$ and $|b\rangle \in \mathcal{B}$ and $\langle a|b\rangle \neq 0$

outcomes $Q(a,b) = \dfrac{\langle b|\hat{\rho}|a\rangle}{\langle b|a\rangle}$

and metric tensor $\boldsymbol{g} = \operatorname{diag}\left[|\langle a|b\rangle|^{-2}\right]_{(a,b)\in\mathcal{A}\times\mathcal{B}}$

→ $P(a,b) = \langle a|b\rangle\langle b|\hat{\rho}|a\rangle$ Kirkwood–Dirac quasiprobabilities

Elementary examples

✓ *s-parametrized phase-space quasiprobabilities*

eight-port homodyning $\hat{\Pi}_\alpha = \frac{|\alpha\rangle\langle\alpha|}{\pi}$ for coherent states $|\alpha\rangle$ with $\alpha \in \mathbb{C}$

outcomes Husimi $Q(\alpha) = \frac{\langle\alpha|\hat{\rho}|\alpha\rangle}{\pi}$

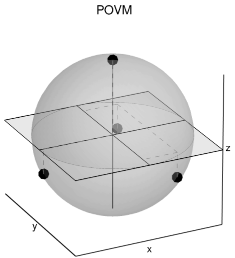
metric tensor via Gaussian kernel $g_{\alpha,\beta} = \pi^{-2} \exp\left(-|\alpha - \beta|^2\right)$

→ $P_\sigma(\alpha) = \pi^\sigma P(\alpha; s)$ as $(s = 2\sigma - 1)$-parametrized quasiprobabilities

e.g., $\sigma = 1/2$ corresponds to $s = 0$ (Wigner function)

Qubit application: informationally complete POVM

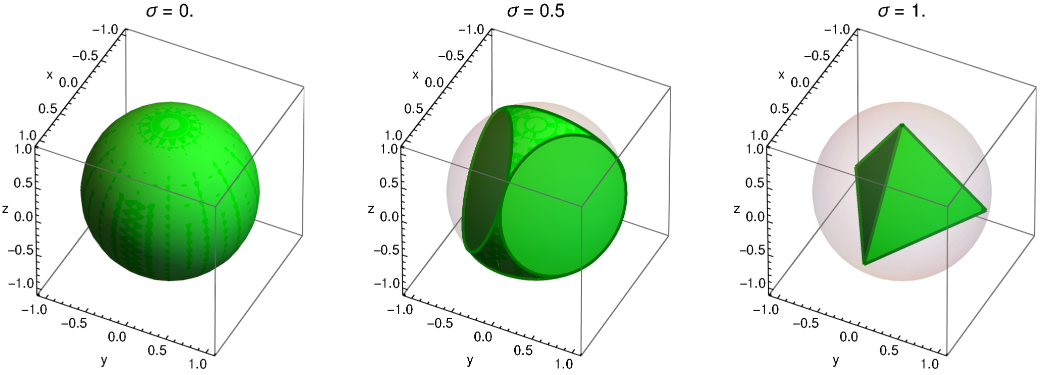
- qubit state $\hat{\rho} \propto \begin{pmatrix} 1+z & x-iy \\ x+iy & 1-z \end{pmatrix}$



- POVM $\hat{\Pi}_k \propto |\psi_k\rangle\langle\psi_k|$ for $k \in \{0,1,2,3\}$

- informationally complete $\vec{Q} \propto \underbrace{\begin{pmatrix} 3 & 3 & 0 & 0 \\ 3 & -1 & 2\sqrt{2} & 0 \\ 3 & -1 & -\sqrt{2} & \sqrt{6} \\ 3 & -1 & -\sqrt{2} & -\sqrt{6} \end{pmatrix}}_{\text{bijective}} \begin{pmatrix} 1 \\ z \\ x \\ y \end{pmatrix}$

Qubit application: informationally complete POVM



Qubit application: incomplete POVM

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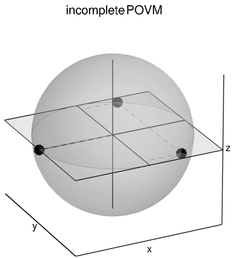
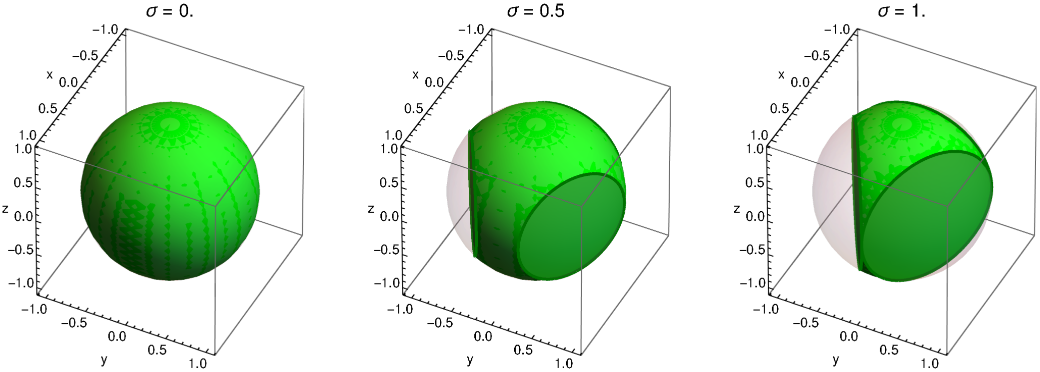


- POVM $\hat{\Pi}_k \propto |\psi_k\rangle\langle\psi_k|$ for $k \in \{0,1,2\}$

- informationally incomplete $\vec{Q} \propto \underbrace{\begin{pmatrix} 2 & 0 & 2 & 0 \\ 2 & 0 & -1 & \sqrt{3} \\ 2 & 0 & -1 & -\sqrt{3} \end{pmatrix}}_{\text{not injective}} \begin{pmatrix} 1 \\ z \\ x \\ y \end{pmatrix}$

Qubit application: incomplete POVM



Qubit application: overcomplete POVM

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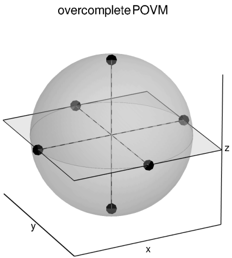
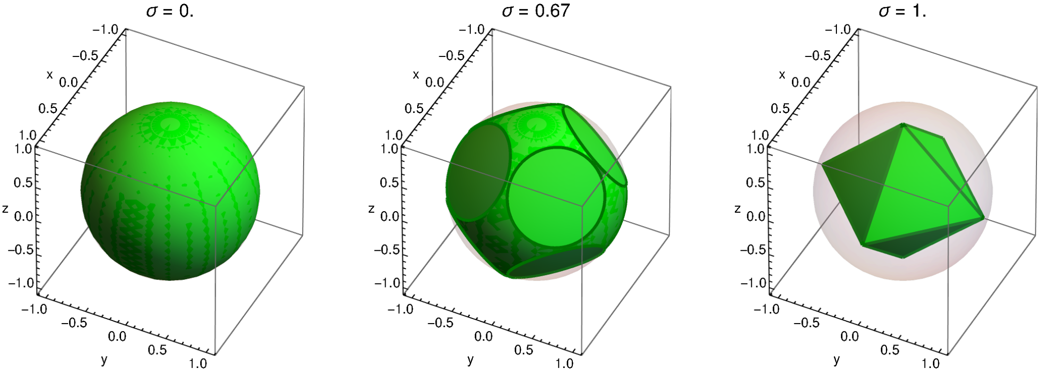


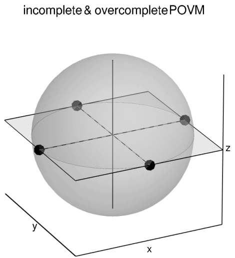
- POVM $\hat{\Pi}_k \propto |\psi_k\rangle\langle\psi_k|$ for $k \in \{0,1,2,3,4,5\}$
- informationally overcomplete $\vec{Q} \propto \underbrace{\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \end{pmatrix}}_{\text{not surjective}} \begin{pmatrix} 1 \\ x \\ y \\ z \end{pmatrix}$

Qubit application: overcomplete POVM



Qubit application: in- & overcomplete POVM

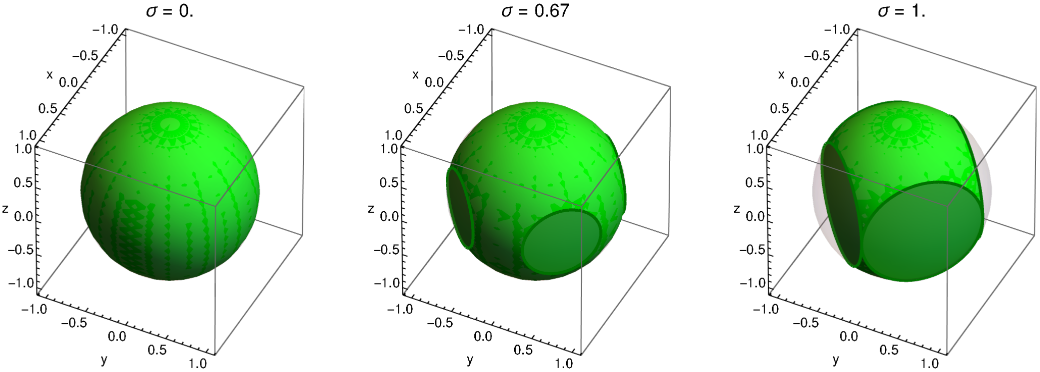
- qubit state $\hat{\rho} \propto \begin{pmatrix} 1+z & x-iy \\ x+iy & 1-z \end{pmatrix}$



- POVM $\hat{\Pi}_k \propto |\psi_k\rangle\langle\psi_k|$ for $k \in \{0,1,2,3\}$

- informationally in- & overcomplete $\vec{Q} \propto \underbrace{\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \end{pmatrix}}_{\text{not surjective \& not injective}} \begin{pmatrix} 1 \\ x \\ y \\ z \end{pmatrix}$

Qubit application: in- & overcomplete POVM



Summary

- ✓ practical problem: quasiprobabilities from measurements
- ✓ including incomplete and overcomplete measurements: still possible to assess nonclassicality
- ✓ partial and pseudo-inversion
- ✓ Kirkwood–Dirac & s-parametrized as special cases
- ✓ proof-of-concept qubit measurements

Safe the date [QPS2026]

Quantum Photonic Spotlight 2026

September 28 – October 2, 2026 | HNF, Paderborn

picture: [QPS2024]

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Many thanks organizers and audience!

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