Efficient Variational Dynamics of Open Quantum Systems In Phase-Space

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Open quantum dynamics

The **Lindblad master equation** unravels the dynamics of a generic quantum system:

Hamiltonian

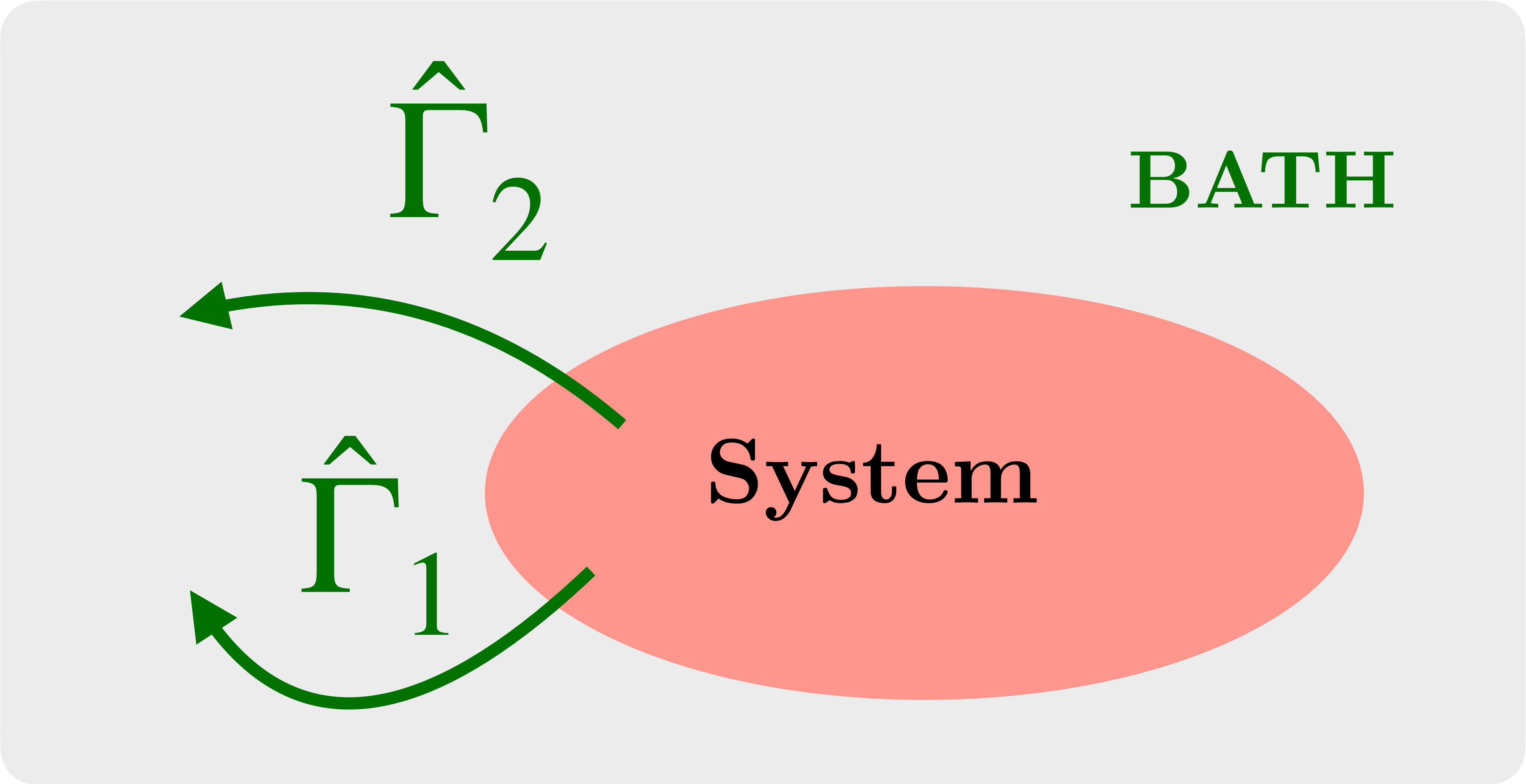
$$\frac{\partial\hat{\rho}}{\partial t} = \mathscr{L}\hat{\rho} = -\,i\left[\hat{H},\hat{\rho}\right] \quad + \quad \sum_j \left(\hat{\Gamma}_j\hat{\rho}\hat{\Gamma}_j - \frac{1}{2}\{\hat{\Gamma}_j^{\dagger}\hat{\Gamma}_j,\hat{\rho}\}\right)$$

Density matrix

COHERENT EVOLUTION

IN-COHERENT EVOLUTION

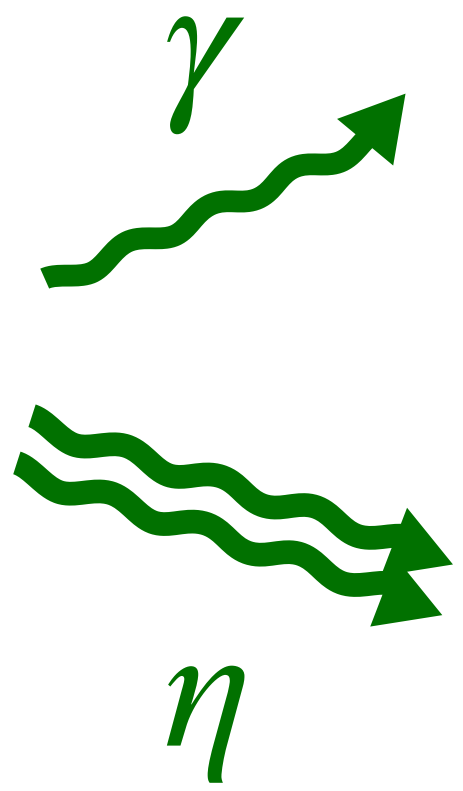
The **incoherent evolution (losses)** is given by the **jump operators** $\hat{\Gamma}_j$



2D Driven-Dissipative Bose-Hubbard

We consider the Hamiltonian of the **2D Driven-Dissipative Bose Hubbard** model with both **single-** and **double-photon** losses γ and η

$$\hat{H} = \sum_{j=1}^{M} \left[-\Delta\, \hat{a}_j^\dagger \hat{a}_j + \frac{U}{2} \hat{a}_j^{\dagger 2} \hat{a}_j^2 + \frac{G}{2} \hat{a}_j^{\dagger 2} + \frac{G^*}{2} \hat{a}_j^2 \right] - \sum_{\langle j,j' \rangle} \frac{J}{z} \left(\hat{a}_j^\dagger \hat{a}_{j'} + \hat{a}_{j'}^\dagger \hat{a}_j \right)$$



Unraveling the whole **quantum dynamics** of a many (at least 100)-modes is **very hard task** especially **in the strongly non-linear regime** ($U/\gamma \gg 1$) **since** **Truncated Wigner, Positive-P** methods **all fail**

Phase-space variational ansatz

In **phase-space** we can reformulate the **Lindblad** master equation as a set of Partial Differential Equations (PDE) governing the dynamics of the Wigner function

$$\frac{\partial \hat{\rho}}{\partial t} = \mathcal{L}\hat{\rho} \quad \Longleftrightarrow \quad \frac{\partial W}{\partial t} = \mathcal{L} W$$

SET OF OPERATORS

SET OF PARTIAL DIFFERENTIAL EQUATION

To solve this, we consider a **variational ansatz (θ)** expressing the **Wigner function** as a **sum** of **Gaussian functions:**

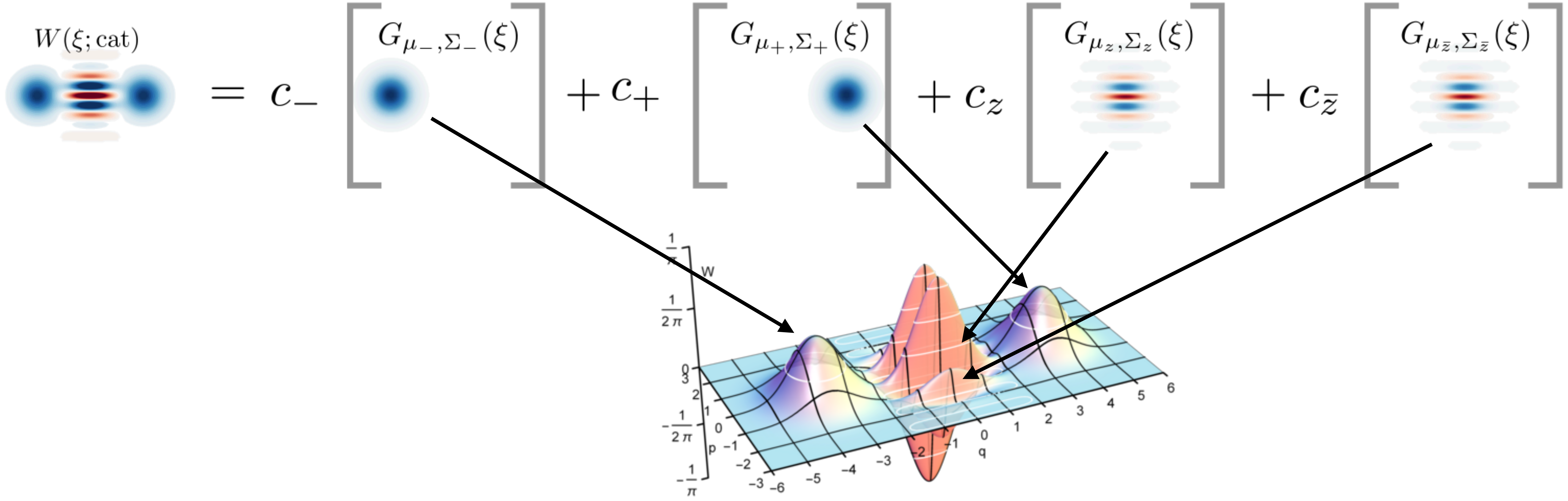
$$\boldsymbol{\theta}_i = (c_i, \boldsymbol{\mu}_i, \Sigma_i)$$

$$W(\boldsymbol{\xi};\boldsymbol{\theta}) = \sum_i G_i(\boldsymbol{\xi};\boldsymbol{\theta}_i)$$

$$G\left(\boldsymbol{\xi};\boldsymbol{\theta}_i\right) = \frac{c_i}{\sqrt{(2\pi)^M \det\left(\Sigma_i\right)}} \exp\left[-\frac{1}{2}\left(\boldsymbol{\xi}-\boldsymbol{\mu}_i\right)\Sigma_i^{-1}\left(\boldsymbol{\xi}-\boldsymbol{\mu}_i\right)\right]$$

Why Gaussians?

Via **multiple complex-centred Gaussians**, highly non-trivial Wigner functions (cat-states, GKP, Fock states...) can be captured*



*Top figure adapted from: *J. Eli Bourassa et al.*, PRX QUANTUM 2, 040315 (2021)

*Bottom figure source: Wikipedia

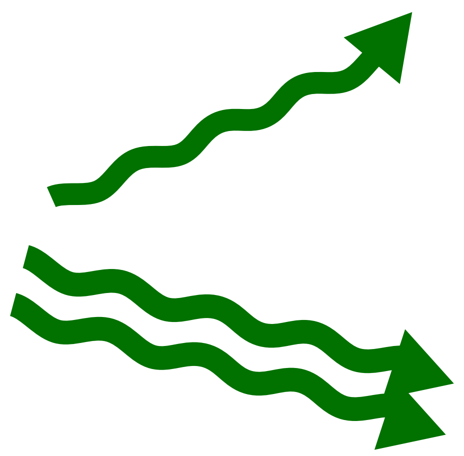
Does it work? Single Kerr Resonator

$$\hat{H} = \frac{U}{2}\hat{a}^{\dagger 2}\hat{a}^{2} + \frac{G}{2}\hat{a}^{\dagger 2} + \frac{G^{*}}{2}\hat{a}^{2} - \Delta\hat{a}^{\dagger}\hat{a}$$

KERR NON-LINEARITY

DOUBLE PHOTON PUMP

DETUNING

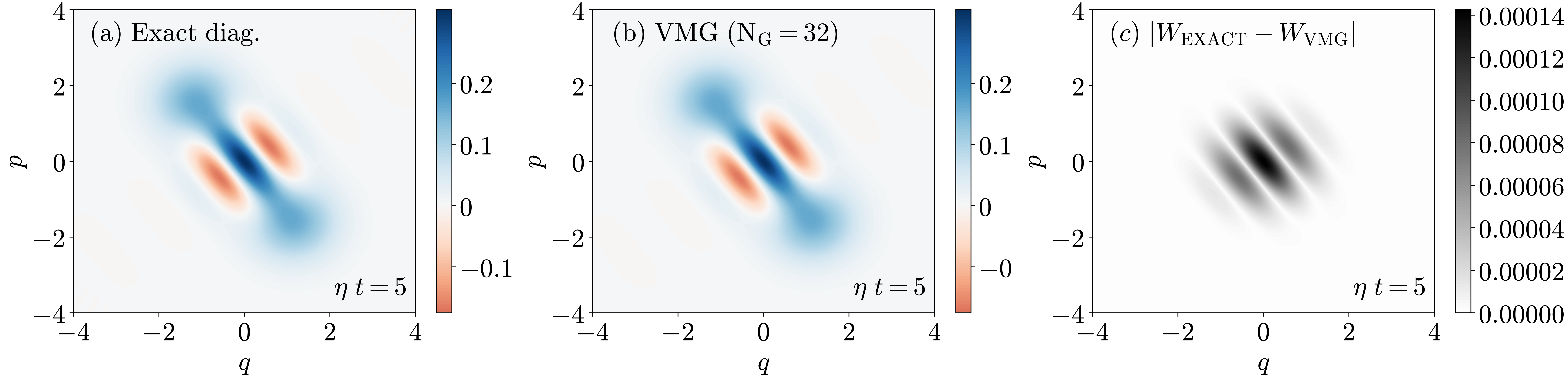


γ SINGLE PHOTON LOSS

η DOUBLE PHOTON LOSS

WIGNER FUNCTION

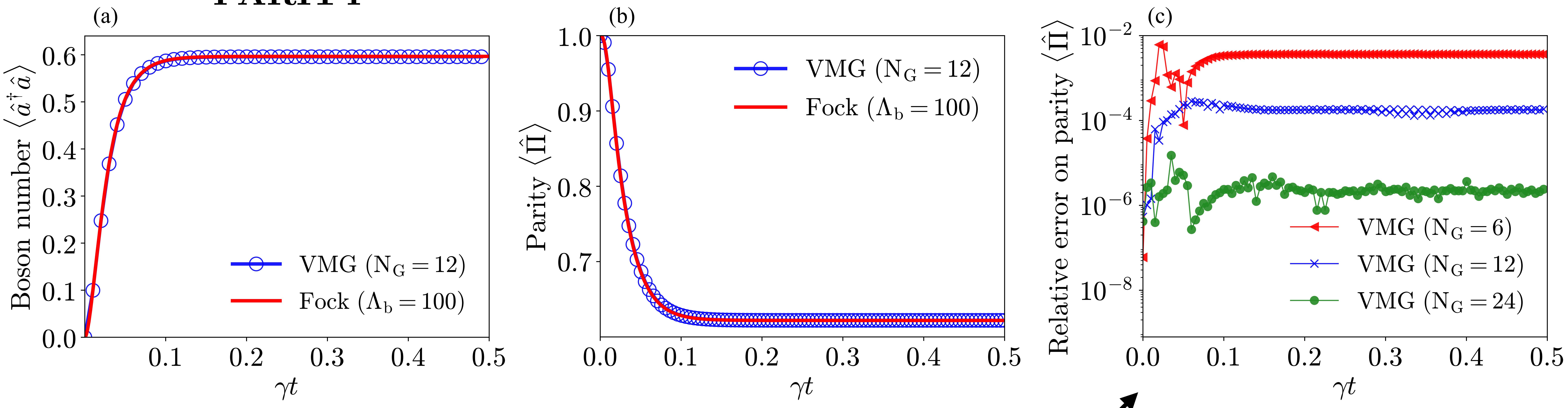
Parameters: $G/\eta = 3 + 3i$, $U/\eta = 1$, $\Delta/\eta = 1$ $\gamma = 0$



Does it work? Single Kerr Resonator

Parameters: $G/\gamma = 0.3 + 0.3i$, $U/\gamma = 0.1$, $\Delta/\gamma = 0.1$ $\eta = 0$

PARITY



Relative error decreases *exponentially* with the number of Gaussians functions

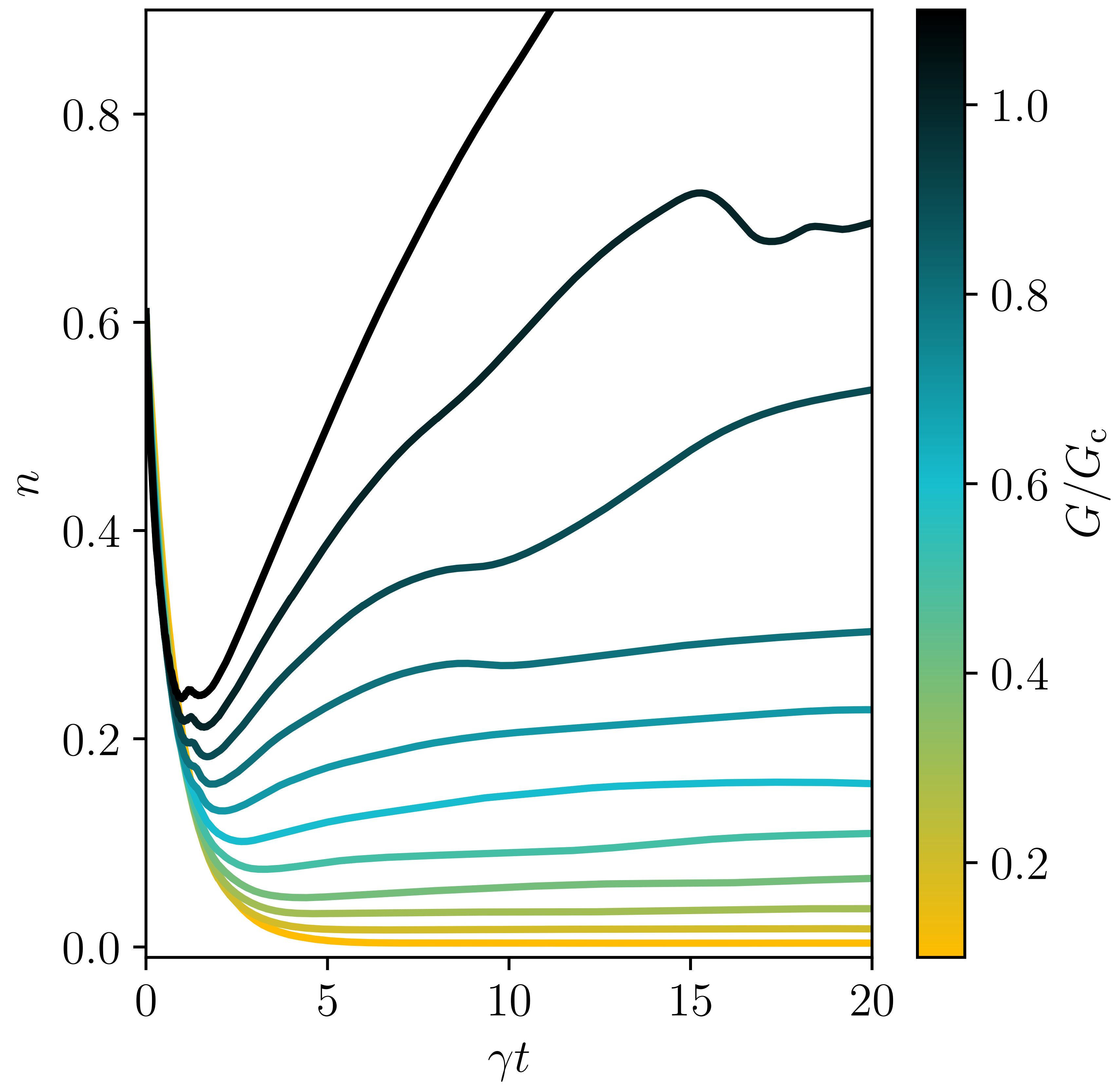
Scale up $\Longrightarrow$ 144 bosonic sites

The Driven Dissipative Bose Hubbard is expected* to exhibit a **second order dynamical phase transition** when tuning the two-photon pump G

To observe it we need to measure the *asymptotic decay rate* λ (or Liouvillian Gap)

$$\lambda = - \lim_{t \to \infty} \frac{1}{t} \Re \left[\ln \langle O(t) \rangle \right]$$

Critical slowing down approaching the critical value of two-photon pumping G



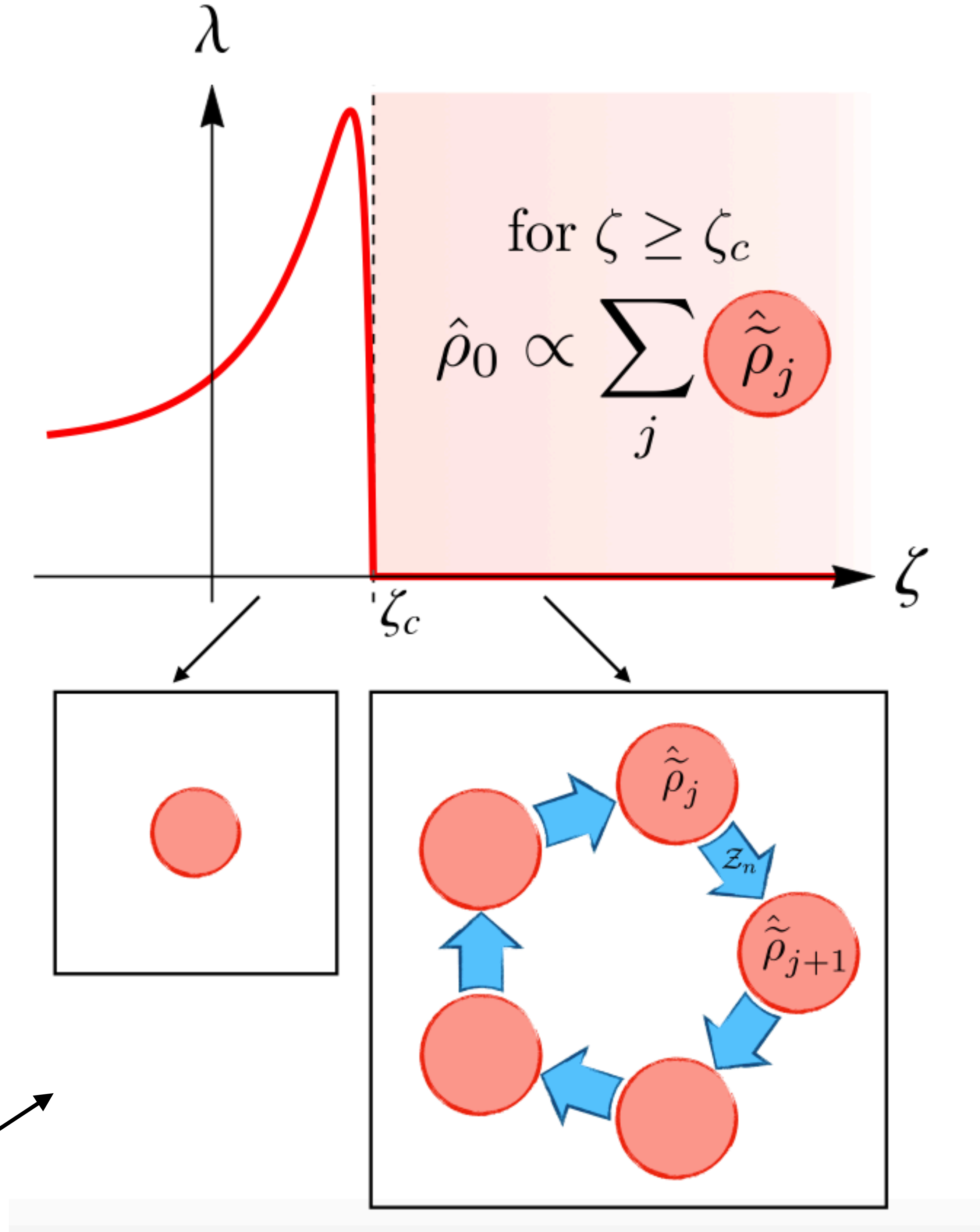
Time evolution for the average photon number ($\hat{n} = \sum_i^M \hat{a}_i^\dagger \hat{a}_i$) for a 12×12 Bose-Hubbard model

R. Rota, F. Minganti, C. Ciuti and V. Savona, PRL, 122, 110405 (2019)

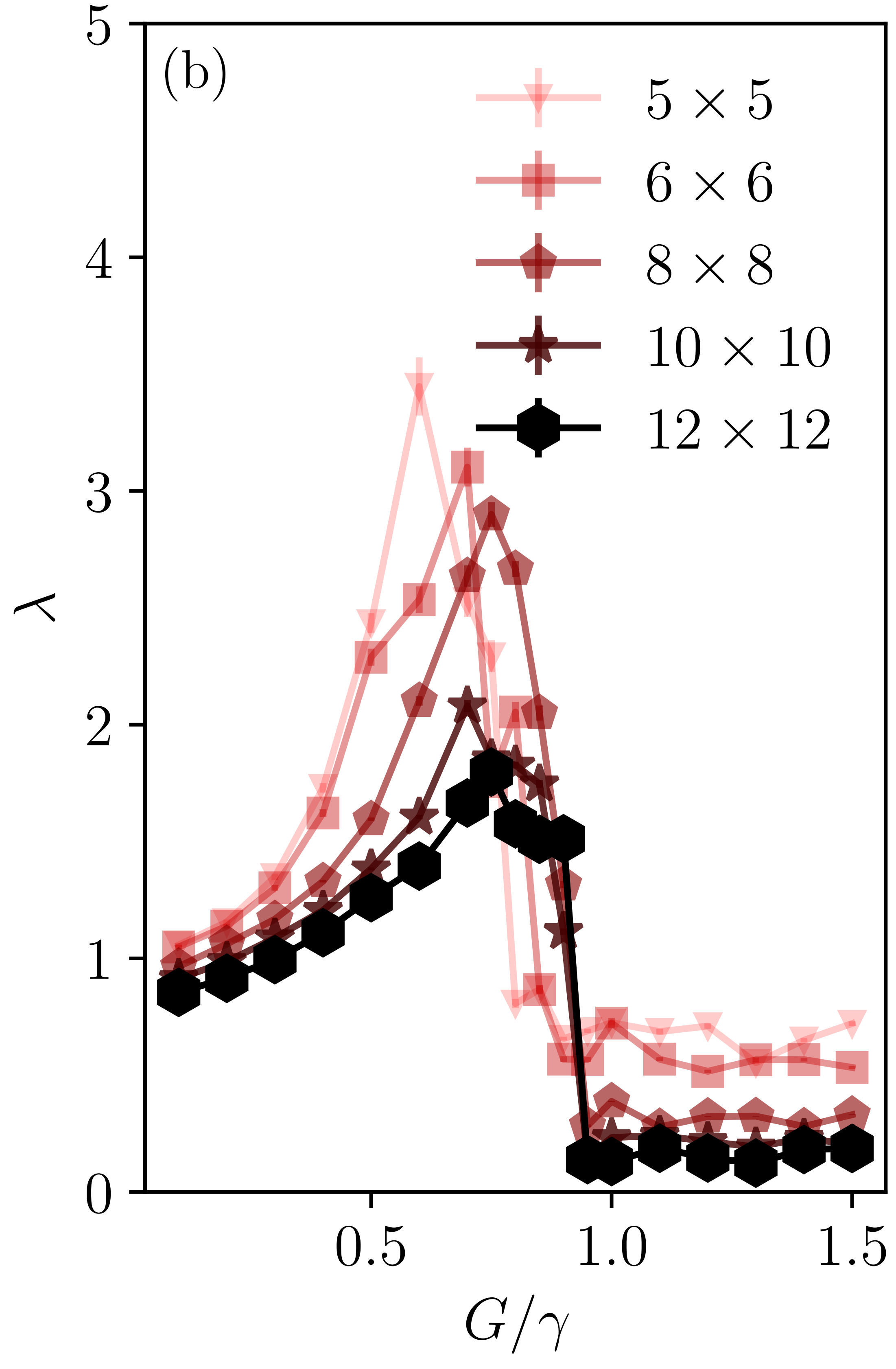
Dynamical dissipative second order phase transition

Parameters: $J/\gamma = 20,$ $U/\gamma = 40,$ $\Delta/\gamma = -20$ $\gamma = \eta$

THEORY

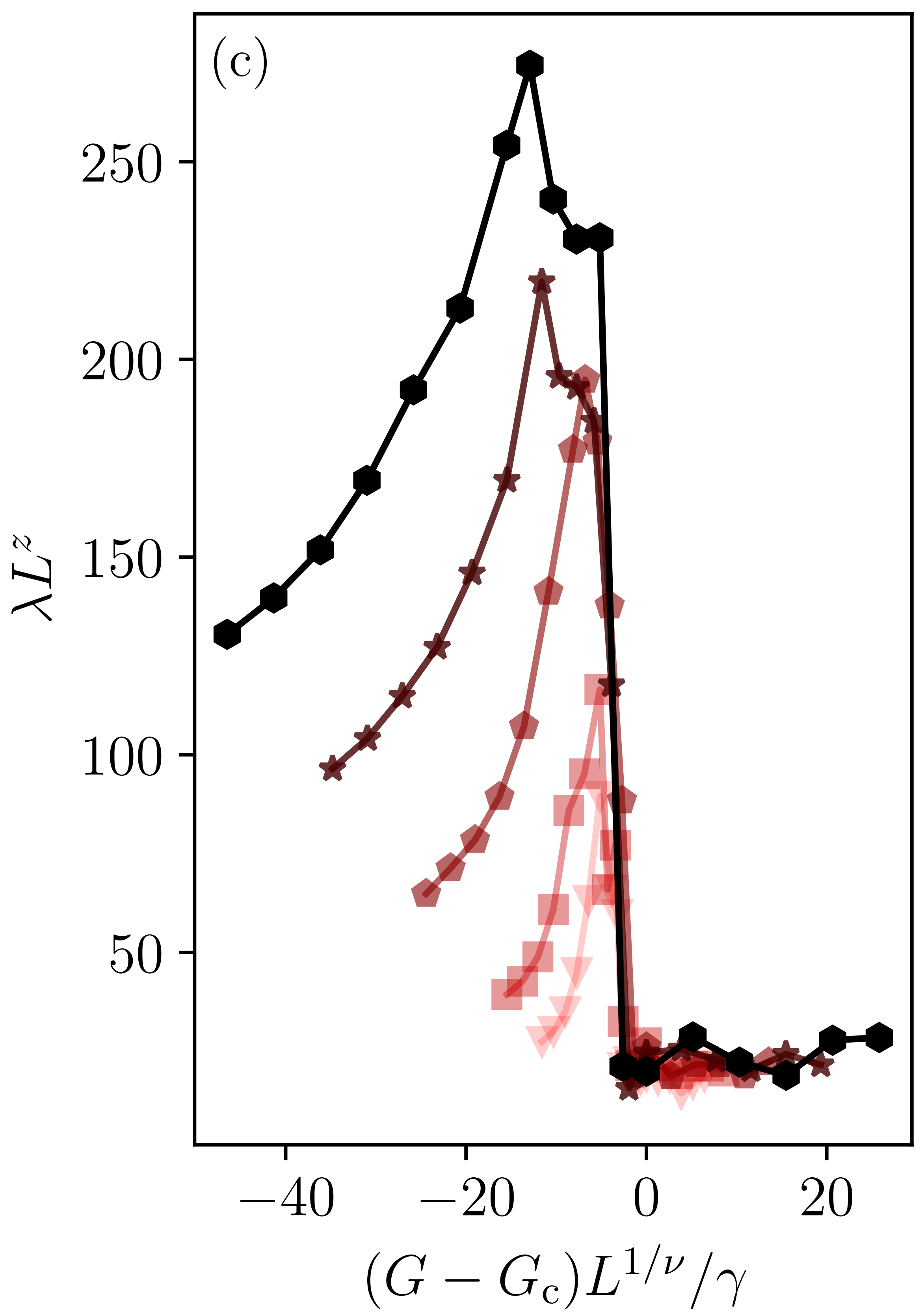


OUR RESULT



Asymptotic decay rate λ

RESCALING AND COLLAPSE



2D Quantum Ising dynamical Critical exponent $z = 2.0235$

*Figure from: *F. Minganti, A. Biella, N. Bartolo and C. Ciuti*, PRA, 98, 042118 (2018)

Thanks for your attention

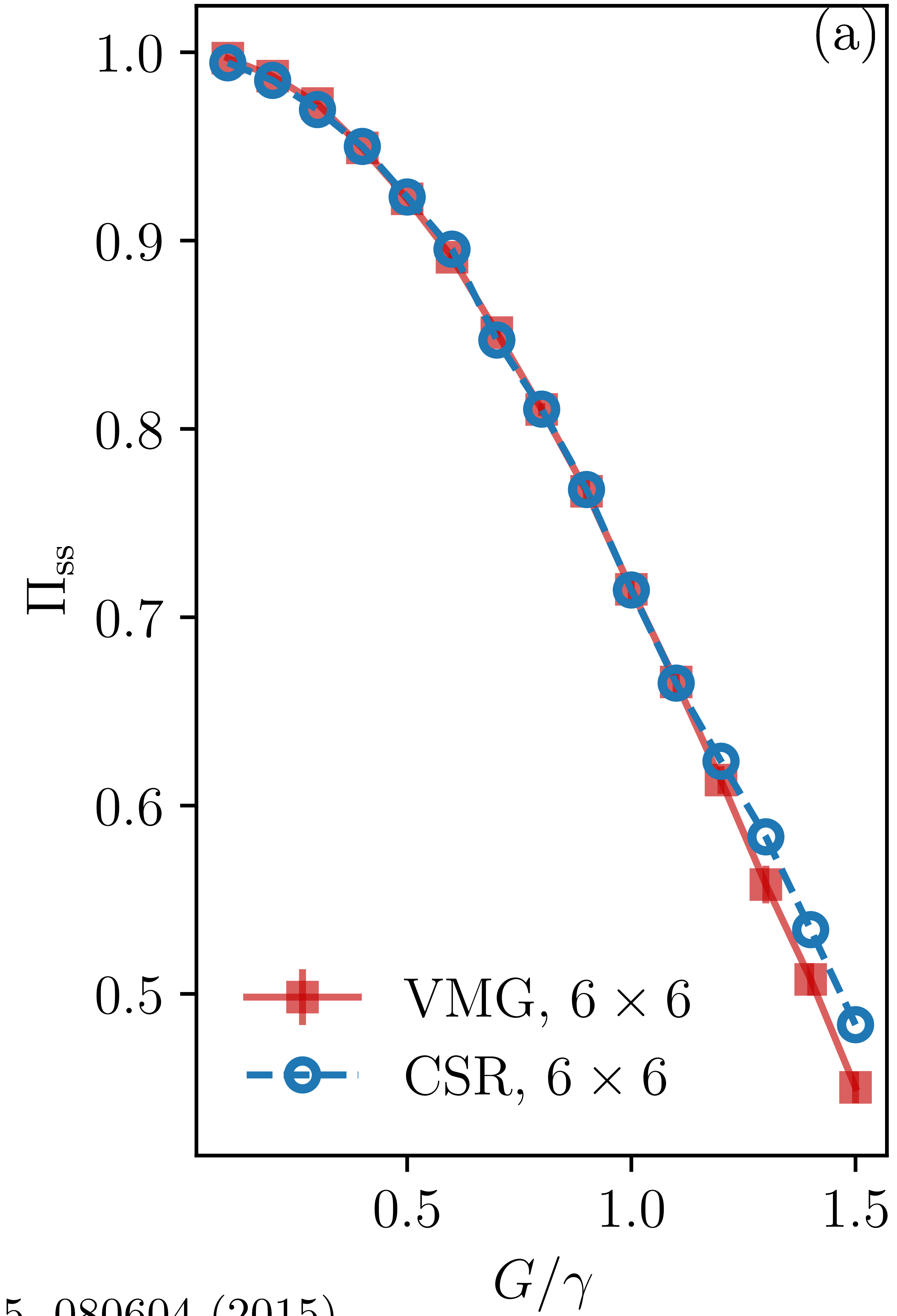
https://arxiv.org/abs/2507.14076

Many mode steady-state benchmark

Driven Dissipative Bose-Hubbard model exhibits a **second order phase transition** in the **photon-number parity operator** $\hat{\Pi} = \exp\left[i\pi \sum_{M} \hat{a}_i^\dagger \hat{a}_i\right]$

$$\hat{H} = \sum_{j=1}^{M} \left[-\Delta\, \hat{a}_j^\dagger \hat{a}_j + \frac{U}{2} \hat{a}_j^{\dagger 2} \hat{a}_j^2 + \frac{G}{2} \hat{a}_j^{\dagger 2} + \frac{G^*}{2} \hat{a}_j^2 \right] - \sum_{\langle j,j' \rangle} \frac{J}{z} \left(\hat{a}_j^\dagger \hat{a}_{j'} + \hat{a}_{j'}^\dagger \hat{a}_j \right)$$

Physical parameters: $\Delta/\gamma = -20, \;\; \eta/\gamma = 1, \;\; U/\gamma = 40, \;\; J/\gamma = 20$



*Corner Space Renormalisation (CSR): *R. Finazzi, A. Le Boité, F. Storme, A. Baksic and C. Ciuti*, PRL, 115, 080604 (2015)

*Data CSR: *R. Rota, F. Minganti, C. Ciuti and V. Savona*, PRL, 122, 110405 (2019)

Variational dynamics for open quantum systems

Let $\rho = \rho(\boldsymbol{\theta}(t))$. **Dirac-Frenkel*** variational principle requires that:

$$\mathrm{tr}\left[(\delta\hat{\rho}(\boldsymbol{\theta}))^\dagger\left(\frac{d}{dt}\hat{\rho}(\boldsymbol{\theta}) - \mathscr{L}\hat{\rho})\right)\right] = 0$$

Which can be rewritten as:
(Assuming real parameters θ)

$$\sum_j \frac{\partial}{\partial\theta_i}\frac{\partial}{\partial\theta'_j}\mathrm{tr}\left(\hat{\rho}(\boldsymbol{\theta})\hat{\rho}(\boldsymbol{\theta}')\right)\Big|_{\boldsymbol{\theta}=\boldsymbol{\theta}'}\dot{\theta}_j = \mathrm{tr}\frac{\partial}{\partial\theta_i}\left(\hat{\rho}(\boldsymbol{\theta})\mathscr{L}\hat{\rho}(\boldsymbol{\theta}')\right)\Big|_{\boldsymbol{\theta}=\boldsymbol{\theta}'}$$

$$\Longrightarrow \quad \boxed{T(\boldsymbol{\theta})\,\dot{\boldsymbol{\theta}} = V(\boldsymbol{\theta})}$$

$$T_{i,j} = \mathrm{Tr}\left[\frac{\partial\rho(\vec{\theta}(t))}{\partial\theta_i}\frac{\partial\rho(\vec{\theta}(t))}{\partial\theta_j}\right] \qquad V_i = \mathrm{Tr}\left[\frac{\partial\rho(\vec{\theta}(t))}{\partial\theta_i}\mathscr{L}(\rho)\right]$$

*$X.$ *Yuan, S. Endo, Q. Zhao, Y. Li, and S. C. Benjamin*, Quantum (2019)

Variational dynamics for open quantum systems

Dirac-Frenkel* variational principle

$$T_{ij}\frac{d}{dt}\boldsymbol{\theta}_j = \boldsymbol{V}_i$$

$$T_{i,j} = \mathrm{Tr}\left[\left(\frac{\partial\rho(\overrightarrow{\theta}(t))}{\partial\theta_i}\right)^{\dagger}\frac{\partial\rho(\overrightarrow{\theta}(t))}{\partial\theta_j}\right]$$

$$V_i = \mathrm{Tr}\left[\left(\frac{\partial\rho(\overrightarrow{\theta}(t))}{\partial\theta_i}\right)^{\dagger}\mathscr{L}(\rho)\right]$$

$\xrightarrow{\text{Phase-space}}$

$$\boldsymbol{T} = \int d^{2M}\boldsymbol{\xi}\Big(\nabla_{\boldsymbol{\theta}}W_{\boldsymbol{\theta}}(\boldsymbol{\xi})\Big)^{T}\otimes\Big(\nabla_{\boldsymbol{\theta}}W_{\boldsymbol{\theta}}(\boldsymbol{\xi})\Big)$$

$$\boldsymbol{V} = \int d^{2M}\boldsymbol{\xi}\Big(\nabla_{\boldsymbol{\theta}}W_{\boldsymbol{\theta}}(\boldsymbol{\xi})\Big)^{T}\mathscr{L}_W(W_{\boldsymbol{\theta}}(\boldsymbol{\xi}))$$

**X. Yuan, S. Endo, Q. Zhao, Y. Li, and S. C. Benjamin*, Quantum (2019)

J. Tosca, F. Carnazza, L.Giacomelli and C. Ciuti, Arxiv 2507.14076 (2025)

Open quantum dynamics in phase-space

In the following we explored a particular class of variational Wigner functions $W_{\boldsymbol{\theta}}$:

Variational Multi-Gaussian (VMG)

$$W_{\boldsymbol{\theta}}(\boldsymbol{\xi}) = \frac{1}{2}\left(\sum_{i=1}^{N_G} G(\boldsymbol{\xi};\boldsymbol{\theta}_i) + c.c.\right) \equiv \sum_{i=1}^{N_G} \mathrm{Re}[G(\boldsymbol{\xi};\boldsymbol{\theta}_i)],$$

DIRAC-FRENKEL

LIOUVILLIAN GRADIENT

$$\boxed{T} \cdot \frac{d}{dt}\boldsymbol{\theta} = \boxed{\boldsymbol{V}}$$

QUANTUM GEOMETRIC TENSOR

$$\boldsymbol{\xi} = (\boldsymbol{q}, \boldsymbol{p})$$

$$\boldsymbol{\theta}_i = (c_i, \boldsymbol{\mu}_i, \Sigma_i)$$

$$G(\boldsymbol{\xi};\boldsymbol{\theta}_i) = \frac{c_i}{\sqrt{(2\pi)^M \det(\Sigma_i)}} \exp\left[-\frac{1}{2}(\boldsymbol{\xi}-\boldsymbol{\mu}_i)\,\Sigma_i^{-1}\,(\boldsymbol{\xi}-\boldsymbol{\mu}_i)\right]$$

$$\boldsymbol{\mu}_i \in \mathbb{C}$$

J. Tosca, F. Carnazza, L.Giacomelli and C. Ciuti, Arxiv 2507.14076 (2025)

Efficient dynamics via Automatic Differentiation

Let's look closer to the form of Quantum Geometric Tensor T for W_θ

QUANTUM GEOMETRIC TENSOR

$$\boxed{T} \cdot \frac{d}{dt}\boldsymbol{\theta} = \boxed{V}$$

LIOUVILLIAN GRADIENT

ANALYTICAL

$$\boxed{T(\boldsymbol{\theta}) = \int d^{2M}\boldsymbol{\xi} \Big(\nabla_{\boldsymbol{\theta}} W_{\boldsymbol{\theta}}(\boldsymbol{\xi}) \Big)^T \otimes \Big(\nabla_{\boldsymbol{\theta}} W_{\boldsymbol{\theta}}(\boldsymbol{\xi}) \Big)} = \sum_{m,n=1}^{N_G} \boxed{\nabla_{\boldsymbol{\theta}} \nabla_{\boldsymbol{\theta}'}} \boxed{\int d^{2M}\boldsymbol{\xi}\ \mathrm{Re}[G(\boldsymbol{\xi};\boldsymbol{\theta})]\mathrm{Re}[G(\boldsymbol{\xi};\boldsymbol{\theta}')]} \Bigg|_{\substack{\boldsymbol{\theta} = \boldsymbol{\theta}_m \\ \boldsymbol{\theta}' = \boldsymbol{\theta}_n}}$$

AUTOMATIC DIFFERENTIATION

Given a variational Multi-Gaussian ansatz $W_{\boldsymbol{\theta}}$, **Automatic Differentiation** provides **fully analytical solution**

Efficient dynamics via Automatic Differentiation

Getting the Liovillian gradient $\boldsymbol{V}$ is just a little but trickier:

$$\boxed{\boldsymbol{V} = \int d^{2M}\boldsymbol{\xi} \;\; (\nabla_{\boldsymbol{\theta}} W_{\boldsymbol{\theta}}(\boldsymbol{\xi}))^T \mathscr{L}_W(W_{\boldsymbol{\theta}}(\boldsymbol{\xi}))}$$

The phase-space Lindbladian operator can always be decomposed as the action of polynomials and derivatives

$$\mathscr{L}_W = \sum_{\mathbf{I}} \mathscr{L}_{W,\mathbf{I}} \qquad \mathscr{L}_{W,\mathbf{I}} \propto \xi_{i_1} \ldots \xi_{i_l} \partial_{\xi_{j_1}} \ldots \partial_{\xi_{j_k}} .$$

ANALYTICAL SOURCE TERM

$$\langle \xi_{i_1} \ldots \xi_{i_l} \partial_{\xi_{j_1}} \ldots \partial_{\xi_{j_k}} \rangle_{\boldsymbol{\theta},\boldsymbol{\theta}'} = \underbrace{\partial_{J_{i_1}} \ldots \partial_{J_{i_l}} \partial_{\tilde{J}_{j_1}} \ldots \partial_{\tilde{J}_{j_k}}}_{\text{AUTOMATIC DIFFERENTIATION}} \mathscr{Z}[\boldsymbol{J}, \tilde{\boldsymbol{J}}] \Big|_{\boldsymbol{J}=0, \tilde{\boldsymbol{J}}=0}$$