Simultaneous Optical Phase and Loss Estimation Revisited: Measurement and Probe Incompatibility

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This work [1] provides a comprehensive analysis of the incompatibility issues in simultaneous quantum estimation of phase and loss in optical interferometry. We distinguish between *probe incompatibility* (the inability of a single probe state to be optimal for all parameters) and *measurement incompatibility* (the impossibility of simultaneously measuring all parameters with optimal precision). Through analytical and numerical methods, including a novel iterative see-saw algorithm for multiparameter optimization, we show that probe incompatibility can be overcome using non-Gaussian states or two-mode Gaussian states with entanglement, while measurement incompatibility remains a fundamental limitation even asymptotically.

INTRODUCTION

Simultaneous estimation of multiple parameters is a key challenge in quantum metrology. In optical settings, phase and loss are two fundamental parameters whose simultaneous estimation is hindered by incompatibilities. This work revisits this problem, focusing on two interferometric scenarios:

1. Single-mode (lossy)
2. Two-mode (one lossy, one reference)

We investigate both Gaussian and non-Gaussian states under a photon number constraint. A third case with two modes, both lossy, was already solved in the literature and results are reviewed for completeness [2, 3].

THEORETICAL FRAMEWORK

We use the following figures of merit:

- **Probe incompatibility [4]:**

$$\mathcal{F}(\rho_{\boldsymbol{\lambda}}) = \frac{1}{d}\sum_{j=1}^{d} \frac{F_{jj}(\rho_{\boldsymbol{\lambda}})}{F^{(\max)}_{\lambda_j}} \leq 1$$

- **Measurement incompatibility [5]:**

$$\mathcal{R}(\rho_{\boldsymbol{\lambda}}) = \frac{\mathrm{Tr}[F(\rho_{\boldsymbol{\lambda}})^{-1}W]}{C^H(W)}$$

where: F_{ij} are the quantum Fisher information (QFI) matrix elements, and $F^{(\max)}_{\lambda_j}$ is the *single-paramter* QFI optimized over probe states, C^H is the fundamental Holevo Cramér-Rao bound that takes into account measurement incompatibility and W is a weight matrix.

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KEY RESULTS

Probe Incompatibility

- **Single-mode Gaussian states** suffer from strong probe incompatibility: $\mathcal{F} \to \frac{1}{2}$.
- **Non-Gaussian states** (optimized via iterative see-saw algorithm) can achieve $\mathcal{F} \to 1$ asymptotically in both single- and two-mode scenarios.
- **Two-mode Gaussian states** with displacement and entanglement (e.g., bright two-mode squeezed states) can also achieve $\mathcal{F} \to 1$.
- Necessary conditions for probe compatibility: high mean photon number and super-Poissonian statistics.

Measurement Incompatibility

- Even for probe-compatible states, measurement incompatibility persists:

$$\overline{\mathcal{R}}^H \xrightarrow{N\to\infty} \frac{2}{3} \quad \text{(non-Gaussian)}, \quad \frac{1}{2} \quad \text{(Gaussian)}$$

- Practical measurements (photon counting, homodyne) also exhibit trade-offs, e.g., different optimal quadratures for phase vs. loss.

CONCLUSION

Probe incompatibility can be overcome with carefully designed states, but measurement incompatibility remains a fundamental obstacle in simultaneous phase and loss estimation. This work provides a unified view of these limitations and introduces numerical tools for future multiparameter quantum metrology studies.
Data availability: `https://github.com/Matheus-Eiji/incomp_phase_loss`

[1] M. E. Ohno Bezerra, F. Albarelli, and R. Demkowicz-Dobrzanski, Simultaneous optical phase and loss estimation revisited: Measurement and probe incompatibility, J. Phys. A: Math. Theor. **58**, 265303 (2025), arXiv:2504.02893.

[2] S. Ragy, M. Jarzyna, and R. Demkowicz-Dobrzański, Compatibility in multiparameter quantum metrology, Phys. Rev. A **94**, 052108 (2016), arXiv:1608.02634.

[3] R. Nichols, P. Liuzzo-Scorpo, P. A. Knott, and G. Adesso, Multiparameter Gaussian quantum metrology, Phys. Rev. A **98**, 012114 (2018), arXiv:1711.09132.

[4] F. Albarelli and R. Demkowicz-Dobrzański, Probe Incompatibility in Multiparameter Noisy Quantum Metrology, Phys. Rev. X **12**, 011039 (2022), arXiv:2104.11264.

[5] F. Belliardo and V. Giovannetti, Incompatibility in quantum parameter estimation, New J. Phys. **23**, 063055 (2021), arXiv:2102.13417.