KD Quasiprobability Meets Bures Geometry: Predictability–Coherence Duality and Triality

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Abstract

We introduce a *predictability* quantifier rooted in geometry and directly tied to quasiprobability-based coherence. For a state ρ and a projective measurement $\{\Pi_a\}$, we define

$$P(\rho, \{\Pi_a\}) \;=\; D_B^2(\,\Delta_a(\rho)\,\|I/d\,),$$

the squared Bures distance between the *dephased* state $\Delta_a(\rho)$ (i.e. the classical outcome distribution in the chosen basis) and the maximally mixed state I/d. This yields a closed form P that depends only on the diagonal probabilities $p_a = \mathrm{Tr}(\rho\Pi_a)$ and satisfies the desiderata for a resource-like "which-path certainty": continuity in p_a, convexity, invariance under basis permutations, monotone decrease under depolarisation, and invariance under the decoherence channel that erases off-diagonals. Moreover, P admits an *entropic representation* via the Tsallis entropy at $\alpha = \frac{1}{2}$,

$$\frac{P(\rho, \Pi_a)}{P_{\max}} \;=\; 1 \;-\; \frac{S_{1/2}(\Delta_a(\rho))}{S_{1/2}(I/d)},$$

linking operational predictability to measurement uncertainty.

Building on *Kirkwood–Dirac (KD)* quasiprobability, we take the pure-state nonclassical KD coherence [1]

$$C_{KD}^{NCL}\big(|\psi\rangle, \{\Pi_a\}\big) \;=\; -1 \;+\; \sum_a \sqrt{|\langle a|\psi\rangle|^2},$$

show that it is the *complement* of the normalised P for pure states, and extend it to *mixed states* by a convex-roof construction [2],

$$C(\rho;\Pi) \;:=\; \inf_{\{p_k,|\psi_k\rangle\}} \sum_k p_k\, C_{KD}^{NCl}\big(|\psi_k\rangle, \{\Pi_a\}\big) \quad \text{s.t.} \quad \rho = \sum_k p_k\, |\psi_k\rangle\!\langle\psi_k|.$$

The resulting $C(\rho;\Pi)$ is a proper coherence monotone: it is faithful (maximal on the uniformly coherent manifold, zero on incoherent states), convex, non-increasing under decoherence and partial trace, covariant under unitaries, and invariant under unitaries commuting with the measured observable and under basis permutations. This furnishes a KD-anchored and geometrically interpretable coherence that pairs naturally with P.

From these constructions we derive (i) a *wave–particle duality* inequality [3, 4]

$$\frac{C(\rho, \{\Pi_a\})}{C_{\max}} \;+\; \frac{P(\rho, \{\Pi_a\})}{P_{\max}} \;\le\; 1,$$

and (ii) using the thermodynamic bound on Tsallis-$\frac{1}{2}$ entropy production [5] (for equilibrium I/d), a *triality equality* [6]

$$\underbrace{\tilde{P}(\rho, \{\Pi_a\})}_{\text{classical}} + \underbrace{\tilde{C}(\rho, \{\Pi_a\})}_{\text{quantum}} + \underbrace{\tilde{S}_{1/2}(\rho)}_{\text{mixedness}} = 1,$$

which cleanly partitions informational resources into classical predictability, quantum coherence, and entropy-like mixedness. We also give an operational interpretation: in a "quantum guessing game with posterior information," C quantifies the *classically irreducible unpredictability*—the part that remains even when an adversary learns the state's pure-state decomposition [7].

For qubits, we provide numerical experiments corroborating the theory, including bounds via Fuchs–van de Graaf inequalities and relations to purity. The framework ties phase-space/quasiprobability notions (KD) to geometric distances (Bures) and thermodynamic constraints (Tsallis) in a unified, experimentally friendly way, with potential applications to metrology, QRNG, and resource accounting in open-system control.

References

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