Quantum statistical functions

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In classical statistics and probability theory, statistical functions such as the moment-generating function, characteristic function, cumulant-generating function, and second characteristic function play a central role in characterizing the statistical properties of a system under consideration [1–4]. The moment-generating function, for instance, provides a compact way to encode all the moments of a probability distribution, while the characteristic function, being its Fourier transform, is always well-defined, even when the moments do not exist. Their logarithmic counterparts, the cumulant-generating and second characteristic functions, are particularly useful for analyzing the independence and additivity of random variables. These tools are indispensable not only in statistics but also in physics, finding applications in statistical mechanics for calculating partition functions and correlation functions, as well as in quantum field theory where generating functionals provide a complete description of correlation functions in quantum field theory.

Despite the profound success of these concepts in classical and quantum field theory, a consistent and unified framework for such statistical functions in the context of quantum mechanics has been lacking [5–13]. Quantum mechanics introduces two fundamental features that complicate this analogy: the noncommutative nature of observables and the intrinsic probabilistic nature of measurement outcomes. While the expectation value of an observable is a well-defined statistical quantity, a systematic framework that generates all higher-order moments and cumulants in a manner analogous to their classical counterparts remains to be fully developed.

This work aims to fill this gap by introducing a set of *quantum statistical functions* for quantum systems. Our approach departs from a direct analogy with classical probability distributions and instead leverages the concept of state purification, a unique feature of quantum mechanics where a mixed state is represented as a pure state on an enlarged Hilbert space [14]. By defining our statistical functions as expectation values of operator functions with respect to this purified state, we establish a robust and comprehensive framework.

Our main contributions are threefold. First, we define a set of four quantum statistical functions: the quantum moment-generating function, characteristic function, cumulant-generating function, and second characteristic function. We demonstrate that these functions, when differentiated, correctly reproduce standard quantum statistical quantities such as expectation values, variance, and covariance. Second, we show that the multi-variable versions of these functions, when defined with specific operator orderings, are intimately connected to well-known quasiprobability distributions [15–18] like the Kirkwood-Dirac distribution [19–21], Margenau-Hill distribution [22], Wigner distribution [23], and so on. This link highlights the role of noncommutativity of operators in shaping the statistical properties of quantum systems. Third, and perhaps most significantly, we extend our framework to include the concept of pre- and post-selection, a non-classical operation paradigm. By defining conditional quantum statistical functions for a pre- and post-selected system, we show that they naturally yield weak values [24, 25] and weak variance [6, 26, 27] upon differentiation.

This work provides a unified mathematical structure that not only reproduces fundamental statistical measures of quantum mechanics but also elegantly incorporates the non-classical aspects of quasiprobabilities. The framework presented here lays the foundation for a deeper understanding of quantum statistics and could inspire new experimental protocols for characterizing quantum states and processes.

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