A structure theorem for complex-valued quasiprobability representations

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Despite its striking empirical success, the ontology implied by quantum theory remains fundamentally unclear. Foundational investigations have nevertheless yielded remarkable approaches for understanding and applying the theory—even if falling short of their original interpretational aims. This flash talk reports structural results at the intersection of two such approaches: quasiprobability representations and the general probabilistic theories framework.

General probabilistic theories (GPTs) [1] originate from the quest to identify what distinguishes quantum theory within a broader landscape of possible physical theories. This framework has elucidated not only aspects of quantum theory, but also properties of any theory that might supersede it. Moreover, it has also clarified that characteristics sometimes considered uniquely quantum are present in various other classes of theories—a short list includes no-cloning [2], no-broadcasting [3], entanglement [4], and teleportation [2]. From this point of view, quantum theory is merely one in a vast landscape of theories.

A seemingly unrelated topic is that of *quasiprobability distributions*, which originate from Eugine Wigner's seminal attempt to represent quantum theory using probability distributions over phase space [5]. In this approach, quantum-mechanical processes are mapped to processes on a phase-space (or more generally, any classical state space). Because the predictions of any quantum experiment can be fully reproduced in such a manner, we refer to this mapping as a *representation* of quantum theory.

If such a representation requires violations of standard axioms of probability [6]—such as allowing negative or nonreal values in the distributions μ_ρ—we call it a *quasiprobability representation*, otherwise we call it a *probability representation*. Formally, such representations can be characterized as *frame representations* [7–10]. Various representations of quantum theory are known, and most commonly one considers *real-valued* representations—i.e., those in which the quasiprobabilities are real-valued functions that may be negative.

Unsurprisingly, quantum theory is not the only theory for which one can meaningfully define a quasiprobability representation. In particular, this notion extends naturally to GPTs. Schmid et al. [11], building on Refs. [12–14], developed a framework for *real-valued finite dimensional* quasiprobability representations that not only captures the structural features of GPTs but also emphasizes their compositional character (see Fig. 1). Most GPTs admit of a diagrammatic representation with an associated compositional calculus, typically formalized as a *category* [15], in which physical systems and processes are represented by diagrams. A quasiprobability representation in this setting is then a map that takes diagrams from a GPT to corresponding diagrams in a target theory defined in terms of quasistochastic matrices.

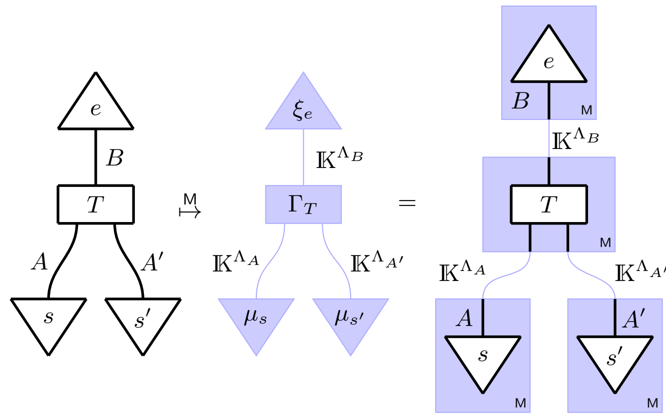

FIG. 1. **Quasiprobability representation of a general probabilistic theory (GPT).** (Left) In a diagrammatic perspective, a physical theory defines a compositional structure for generic systems (denoted $A, B, \dots$) and processes between them (denoted $s, e, T, \dots$). (Right) A quasiprobability representation is a map M that assigns to each system A a set of relevant variables Λ_A taking values in $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, and to each process, a quasistochastic element—such as a quasiprobability distribution μ_s, a response function ξ_e, or a quasisubstochastic matrix Γ_T.

Formally, this diagrammatic view captures the compositional structure of a GPT in the language of *process theories*—mathematical structures closely related to symmetric monoidal categories. The target theory of a quasiprobability representation is described using quasisubstochastic matrices: systems correspond to real vector spaces, processes to matrices, states to unnormalized quasiprobability vectors, and so on. From this perspective, such representations correspond to *semi-functors*, as they translate the process-theoretic structure of one theory into that of another while preserving its compositional connectivity.

In Schmid et al. [11], some of us proved a structure theorem for all *diagram-preserving maps*—a subclass of semifunctorial maps that, additionally, preserve identity processes, hence corresponding to what is known as a *functor*. Their framework, together with the structure theorems proved there, can be viewed as a category-theoretic extension of the linear-algebraic program that classifies *linear preservers* [16]. In this program, one studies linear operators on a *structured* matrix space $\mathcal{O} \subseteq \mathrm{Mat}_{\mathbb{K}}(n, m)$ and seeks all maps $T : \mathcal{O} \to \mathcal{O}$ that preserve the set (i.e. $T(\mathcal{O}) = \mathcal{O}$, or in the weaker form $T(\mathcal{O}) \subseteq \mathcal{O}$).

Most results in this research program show that T typically has the form

$$X \mapsto AXB \text{ or } X \mapsto AX^T B, \tag{1}$$

for $A, B \in \mathcal{O}$. The structure theorem of Ref. [11] states that

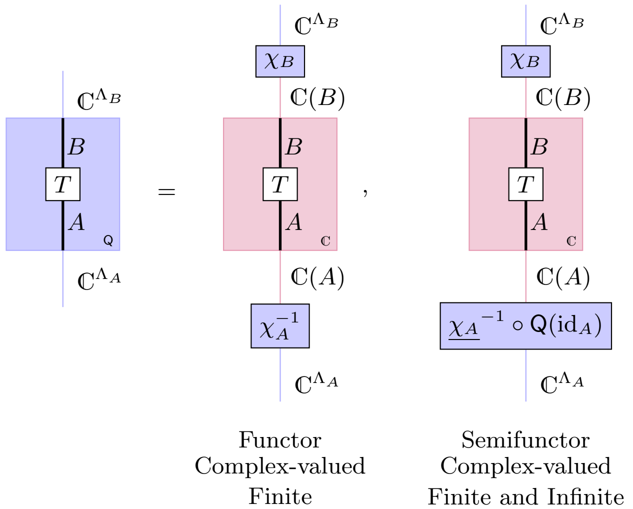

FIG. 2. **Illustration of the main results.** Any linearity-preserving, empirically adequate, complex-valued quasiprobability representation Q of a generic transformation $T : A \to B$ in a tomographically local, finite-dimensional GPT can be written as $\mathsf{Q}(T) = \chi_B \circ \mathbb{C}(T) \circ \chi_A^{-1}$, provided Q is functorial and maps the GPT to finite-dimensional spaces. If Q is only semi-functorial, then $\mathsf{Q}(\mathrm{id}_A)$ is an idempotent and $\mathsf{Q}(T) = \chi_B \circ \mathbb{C}(T) \circ \underline{\chi_A}^{-1} \circ \mathsf{Q}(\mathrm{id}_A)$, where χ_A is injective and $\underline{\chi_A}$ denotes its invertible surjective corestriction. In these formulas, $\mathbb{C}$ denotes the complexification functor.

any real-valued finite dimensional quasiprobability functorial representation M which preserves a few structures relevant for the GPT is equivalently implemented by system-wise maps $A \mapsto \chi_A : A \to \mathsf{M}(A)$ and acts by conjugation on transformations:

$$\mathsf{M}(T) = \chi_B \circ T \circ \chi_A^{-1} \quad (2)$$

for every transformation $T : A \to B$ in the GPT.

Recently, a particularly important class of *complex-valued* quasiprobability representations has attracted significant attention: the Kirkwood–Dirac (KD) quasiprobability distributions. First introduced by Kirkwood in the 1930s [17] and later rediscovered by Dirac [18], these distributions have undergone a revival of interest [19–22]. Most relevantly for our discussion, they have been shown to lift to faithful complex-valued diagram-preserving representations of all of quantum theory [23].

Kirkwood–Dirac representations—and more generally, complex-valued quasiprobability distributions—lie outside the scope of the formalism and structure theorem of Ref. [11]. This motivates the need for a broader understanding of the structural features of complex-valued representations of quantum theory in particular, and more broadly of any GPTs. The present flashtalk reports on work which takes a step in that direction. Our main contribution is to establish a structure theorem showing that any complex-valued quasiprobability representation of a finite-dimensional, tomographically local GPT has a simple and constrained mathematical form.

Our results are illustrated in Fig. 2 and are obtained from the following sequence of ideas. Since we consider complex-valued representations, we begin by describing in detail the *complexification* of real vector spaces, and linear maps between them. We then show how complexification can be viewed as a category-theoretic construction, proving that it yields a strong monoidal functor. Equipped with the complexification functor, we then prove a structure theorem for semi-functors from tomographically local GPTs to the process theory of *complex* vector spaces. We prove a structure theorem for complex-valued quasiprobability representations (which in our formalism are a subset of all possible semi-functors) of tomographically local GPTs, extending the results from Ref. [11]. Note that unlike in Ref. [11], our results apply equally as well to both finite and infinite dimensional representations.

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